Quasi-isometries and hyperbolic spaces

Exercice 1 Let G be a group acting on a connected topological space X and assume there is an open subset $\mathrm{U} \subset \mathrm{X}$ such that $\mathrm{G}(\mathrm{U}) = \mathrm{X}$. Prove that

$$\mathrm{S} = \{g \in \mathrm{G},\ g(\mathrm{U}) \cap \mathrm{U} \neq \varnothing\}$$

is a generating set for G.

Exercice 2 Prove that $\mathbb{Z}$ and $\mathbb{R}$ are quasi-isometric.

Exercice 3 Prove that the binary relation "to be quasi-isometric" on the collection of metric spaces defines an equivalence relation.

Exercice 4 Let X and Y be two metric spaces. Prove that they are quasi-isometric if, and only if, they contain cobounded subsets $\mathrm{X}' \subset \mathrm{X}$ et $\mathrm{Y}' \subset \mathrm{Y}$ that are quasi-isometric.

Exercice 5 Let X and Y be two metric spaces. A quasi-isometric correspondence is given by two constants $\lambda \geq 1$ et $c > 0$ and a binary relation $\mathcal{R} \subset \mathrm{X} \times \mathrm{Y}$ such that

1. For all $x \in \mathrm{X}$, there exists $y \in \mathrm{Y}$ such that $x\mathcal{R}y$; for all $y \in \mathrm{Y}$, there exists $x \in \mathrm{X}$ such that $x\mathcal{R}y$;
2. If $x\mathcal{R}y$ and $x'\mathcal{R}y'$, then $\frac{1}{\lambda}d_{\mathrm{X}}(x,x') - c \leq d_{\mathrm{Y}}(y,y') \leq \lambda d_{\mathrm{X}}(x,x') + c$.

Prove that X and Y are quasi-isometric if and only if there exists a quasi-isometric correspondence.

Exercice 6 This exercise is devoted to proving that, for any $\varepsilon > 0$, any length space X is $(1+\varepsilon, 2)$-quasi-isometric to a graph Γ, in which each edge is isometric to the segment $[0,1]$. Let $\delta > 0$.

1. Prove that there exists a maximal collection $\mathcal{B}_\delta$ of balls of radius δ that are pairwise disjoint in X. Prove that, for any $x \in \mathrm{X}$, there exists a ball $\mathrm{B} = \mathrm{B}(c,\delta) \in \mathcal{B}_\delta$ such that $d(x,c) < 2\delta$.
2. Let us consider the graph $\Gamma = (\mathrm{V}, \mathrm{E})$, where $\mathrm{V} = \mathcal{B}_\delta$ and $(\mathrm{B}, \mathrm{B}') \in \mathrm{E}$, $\mathrm{B} = \mathrm{B}(c,\delta)$, $\mathrm{B}' = \mathrm{B}(c',\delta)$, if $d(c,c') \leq 1$. Prove that the map $\mathrm{B} = \mathrm{B}(c,\delta) \in \Gamma \mapsto c \in \mathrm{X}$ defines a $(1+\varepsilon, 2)$-quasi-isometry, where $\varepsilon = \mathrm{O}(\delta)$.

Exercice 7 Let G be a finitely generated group.

1. Prove that if $\mathrm{H} < \mathrm{G}$ is a finite index subgroup, then it is also finitely generated and it is quasi-isometric to G. To this purpose, we may let H act on the Cayley graph of a finite generating set of G.
2. Prove that if N is a finite normal subgroup of G then G and G/N are quasi-isometric.

Exercice 8 Let T_n be the regular infinite tree where each vertex has n edges to it, $n \geq 3$. Let us endow T_n with the length distance that makes each edge isometric to $[0,1]$.

1. Let us color the edges of T_3 with three colors a, b and c in such a way that each vertex is the endpoint of an edge of each color. Let us consider the equivalence relation $x \sim x'$ if $\{x, x'\}$ are the ends of an edge of color a.
   - *a*) Prove that $\mathrm{T}_3/\sim$ is isomorphic to T_4.
   - *b*) Prove that T_3 and T_4 are quasi-isometric.
2. Prove that T_m and T_n are quasi-isometric if $m, n \geq 3$.
3. Deduce that the free groups $\mathbb{F}_m$ and $\mathbb{F}_n$ are quasi-isometric as soon as $m, n \geq 2$.

Exercice 9 Let g be an isometry of a metric hyperbolic space X.

1. Prove that the following limit exists and is independent of the point $x \in \mathrm{X}$

$$\tau(g) = \lim_{n\to\pm\infty} \frac{d(x, g^n(x))}{n} = \inf_{n\geq 1} \frac{d(x, g^n(x))}{n}.$$

We may apply Fekete's lemma on subadditive sequences.

2. Prove that the following properties are equivalent.

a) g is a loxodromic isometry ;

b) for all $x \in \mathrm{X}$, $\{g^n(x)\}_n$ is a quasigeodesic;

c) $\tau(g) > 0$ holds.

Exercice 10 Let X be a hyperbolic geodesic proper metric space, let $a \in \partial\mathrm{X}$ and let g be an isometry that fixes the point a.

1. Prove that $\mathrm{B}(g) = \lim \frac{1}{n}\beta_a(x, g^n x)$ is well-defined, and does not depend on the point $x \in \mathrm{X}$.

2. Prove the existence of a constant $\mathrm{C} \geq 0$, $\mathrm{C} = \mathrm{C}(\delta)$, such that, for all $x \in \mathrm{X}$, $|\beta_a(gx, x) - \mathrm{B}(g)| \leq \mathrm{C}$ holds.

3. Prove that $\mathrm{B}(g) \neq 0$ if, and only if g is loxodromic.

4. Prove that if G is a parabolic group with fixed point a then, there exists $\mathrm{C} \geq 0$ such that, for any $g \in \mathrm{G}$ and any $x \in \mathrm{X}$, $|\beta_a(x, gx)| \leq \mathrm{C}$ holds.

Exercice 11 Let G be a group acting geometrically on a hyperbolic geodesic proper space X.

1. Prove that any element $g \in \mathrm{G}$ is either finite or loxodromic.

2. Prove that its action is cocompact on the set $\Theta(\partial\mathrm{X})$ of distinct triples.

3. Prove that any point of $\partial\mathrm{X}$ is conical for G.

Exercice 12 Let G be a group generated by a finite S and let Z be its Cayley graph. We assume that Z is hyperbolic in the sense of Gromov.

1. Prove that there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, there is a distance d_ε on G and a constant $\mathrm{C} \geq 1$ such that $(1/\mathrm{C})\mathrm{e}^{-\varepsilon(x|y)_\mathrm{e}} \leq d_\varepsilon(x, y) \leq \mathrm{C}\mathrm{e}^{-\varepsilon(x|y)_\mathrm{e}}$ holds for all $x, y \in \mathrm{G}$.

We fix $\varepsilon \in (0, \varepsilon_0)$ as above.

2. Let $\overline{\mathrm{G}}_\varepsilon$ be the completion of $(\mathrm{G}, d_\varepsilon)$ and prove that $\overline{\mathrm{G}}_\varepsilon \setminus \mathrm{G}$ is homeomorphic to the Gromov boundary $\partial_\mathrm{G}\mathrm{Z}$ of Z.

3. Set $f(r) = \mathrm{e}^{-\varepsilon r}$ for $r \geq 0$.

a) Check that f is a Floyd gauge and that there exists a constant $\mathrm{C}_f \geq 1$ such that, for any edge (x, y), one has $(1/\mathrm{C}_f)\mathrm{e}^{-\varepsilon(x|y)_\mathrm{e}} \leq f(d(\mathrm{e}, \{x, y\})) \leq \mathrm{C}_f\mathrm{e}^{-\varepsilon(x|y)_\mathrm{e}}$ holds for all $x, y \in \mathrm{G}$.

b) Let $x, y \in \mathrm{Z}$, and let γ be a geodesic segment in Z joining x and y. Prove that there exists a constant $\mathrm{C}_+ \geq 1$ that only depends on the hyperbolicity constant of Z such that $\ell_f(\gamma) \leq \mathrm{C}_+ d_\varepsilon(x, y)$ holds. One may consider the associated tripod $\{\mathrm{e}, x, y\}$ to estimate the Floyd-length of γ.

c) Let us consider a curve γ joining x and y that is not necessarily geodesic. Prove that there exists a constant $\mathrm{C}_- \geq 1$ such that $\ell_f(\gamma) \geq (1/\mathrm{C}_-)d_\varepsilon(x, y)$.

d) Prove that the Floyd boundary $\partial_f\mathrm{Z}$ and the Gromov boundary $\partial_\mathrm{G}\mathrm{Z}$ are bi-Lipschitz equivalent.