General properties of convergence groups

Exercice 1 Let G be a closed convergence group acting on a metrizable compact space X. Let K, L be two disjoint compact subsets of X. Prove that the set E of non-loxodromic elements $g \in \mathrm{G}$ such that $g(\mathrm{K}) \cap \mathrm{K} \neq \varnothing$ et $g(\mathrm{L}) \cap \mathrm{L} \neq \varnothing$ is compact.

Exercice 2 Let G be a closed Abelian and non-compact convergence group acting on a metrizable compact space X.

1. Prove that the set of fixed points of a given element $h \in \mathrm{G}$ is G-invariant.

2. Prove that one of the following holds

- Either there is a unique fixed point p under G. All of its elements are either elliptic or parabolic.
- Or there is a G-invariant set formed of two points $\{a, b\}$. There is a properly discontinuous and cocompact action of $\mathbb{Z}$ on $\mathrm{X} \setminus \{a, b\}$ and on G. All elements of G are either elliptic or loxodromic.

Exercice 3 Let G be a locally compact convergence group acting on a metrizable compact space X. Let H be a normal subgroup of G.

1. Prove that if H is compact, then $\Lambda_{\mathrm{H}} = \varnothing$ and H acts trivially on Λ_{G}.

2. Prove that if H is non compact then either Λ_{H} is a single point and G admits a fixed point or $\Lambda_{\mathrm{H}} = \Lambda_{\mathrm{G}}$.

3. Assume that G is of general type. Prove that there exists a unique maximal normal compact subgroup W, and there exists a compact space Y and a continuous, onto, equivariant map $p : \mathrm{X} \to \mathrm{Y}$ such that the restriction of p to Λ_{G} is a homeomorphism onto its image and the action of G on Y yields a faithful convergence action of G/W.

For the remaining exercises, we consider the following setup.

Let X, Y be two metrizable compact spaces. Let $\mathrm{W}_{\mathrm{K},\mathrm{V}} = \{h \in \Phi,\ h(\mathrm{K}) \subset \mathrm{V}\}$, where $\mathrm{K} \subset \mathrm{X}$ is compact and $\mathrm{V} \subset \mathrm{Y}$ is open. Set

$$\mathrm{Z} = (\mathrm{X} \times \mathrm{Y}) \sqcup \mathrm{Homeo}(\mathrm{X}; \mathrm{Y})$$

with the following topology. Neighborhoods of elements $\phi \in \mathrm{Homeo}(\mathrm{X}; \mathrm{Y})$ are of the form $\mathrm{W}_{\mathrm{K},\mathrm{V}}$ where $\phi(\mathrm{K}) \subset \mathrm{V}$. For points $(x, y) \in \mathrm{X} \times \mathrm{Y}$, a basis of neighborhoods will be given by sets of the form $(\mathrm{U} \times \mathrm{V}) \sqcup \mathrm{W}_{\mathrm{X}\setminus\mathrm{U},\mathrm{V}}$. We shall also consider a closed subset $\Phi \subset \mathrm{Homeo}(\mathrm{X}; \mathrm{Y})$ and the corresponding space $\mathrm{Z}_\Phi = (\mathrm{X} \times \mathrm{Y}) \sqcup \Phi$ with the induced topology.

Exercice 4 We use the above notation.

1. Prove that Z is Hausdorff.

2. Show that a sequence $(\phi_n)_n$ in Homeo(X; Y) tends to $(x, y) \in \mathrm{X} \times \mathrm{Y}$ in Z if and only if (ϕ_n) is a convergence sequence with base (x, y).

3. Show that Z_Φ is compact if and only if Φ has the convergence property, i.e., any $\Phi' \subset \Phi$ is either relatively compact or contains a convergence sequence.

Exercice 5 We assume that Φ satisfies the convergence property. Let $\widehat{\Lambda} = \overline{\Phi} \cap (\mathrm{X} \times \mathrm{Y})$ where $\overline{\Phi}$ is the closure in Z.

1. Prove that $\widehat{\Lambda}$ is compact.

2. Show that $(x, y) \in \widehat{\Lambda}$ if and only if there is a convergence sequence $(\phi_n)_n$ tending to (x, y) in Z.

Exercice 6 Let us consider the action de Homeo(X) $\times$ Homeo(Y) on Homeo(X; Y) given by ρ_c : $(h_\mathrm{X}, h_\mathrm{Y}) \cdot \phi = h_\mathrm{Y} \circ \phi \circ h_\mathrm{X}^{-1}$ and assume $\Phi \subset$ Homeo(X; Y) satisfies the convergence property. For any $h = (h_\mathrm{X}, h_\mathrm{Y}) \in$ Homeo(X) $\times$ Homeo(Y), write $\widehat{h} = \rho_c(h)$.

1. Prove that, for any $h = (h_\mathrm{X}, h_\mathrm{Y}) \in$ Homeo(X) $\times$ Homeo(Y), the set $\widehat{h}(\Phi)$ also satisfies the convergence property and $\widehat{h}$ defines a homeomorphism $\widehat{h} : \mathrm{Z}_\Phi \to \mathrm{Z}_{\widehat{h}(\Phi)}$.

2. Prove that the stabilizer H_Φ of Φ in Homeo(X) $\times$ Homeo(Y) acts continuously on Z_Φ.

Exercice 7 Let $\Phi \subset$ Homeo(X, Y) be closed with the convergence property.

1. Show that the map $\phi \mapsto \phi^{-1}$ induces an equivariant homeomorphism $\iota : \mathrm{Z}_\Phi \to \mathrm{Z}_{\iota(\Phi)}$ such that $\iota(x, y) = (y, x)$.

2. Prove that $\iota(\widehat{\Lambda}_\Phi) = \widehat{\Lambda}_{\iota(\Phi)}$.

3. Deduce that if (ϕ_n) is a convergence sequence of base (x, y) then (ϕ_n^{-1}) is a convergence sequence of base (y, x).

Exercice 8 We now assume that X = Y and that Φ = G is a convergence group. Define on $\mathrm{Z}_\mathrm{G} = (\mathrm{X} \times \mathrm{X}) \cup \mathrm{G}$ the equivalence relation generated by the relation $(x, y) \sim (x', y')$ if $y = y'$ for $(x, y), (x', y') \in \mathrm{X} \times \mathrm{X}$. Let $\mathrm{X}_\mathrm{G} = \mathrm{Z}_\mathrm{G}/\sim$ be endowed with the quotient topology and $p : \mathrm{Z}_\mathrm{G} \to \mathrm{X}_\mathrm{G}$ be the canonical projection.

1. Prove that $\mathrm{X}^2/\sim$ is homeomorphic to X.

2. Prove that neighborhoods of $\mathrm{X} \times \{y\}$ in Z_G are satured for all $y \in \mathrm{X}$ and that they define a basis of neighborhoods of $p(\mathrm{X} \times \{y\})$.

3. Prove that X_G is compact.

4. Show that the restriction $p : \mathrm{G} \to \mathrm{X}_\mathrm{G}$ is a homeomorphism on its image.

5. Prove that $\Lambda_\mathrm{G} = p(\widehat{\Lambda}_\mathrm{G})$ and $\Omega_\mathrm{G} = \mathrm{G} \sqcup (\mathrm{X} \setminus \Lambda_\mathrm{G})$.