Quasi-isometric rigidity of the class of convex-cocompact Kleinian groups

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Abstract. We provide a streamlined proof that the class of convex-cocompact Kleinian groups are quasi-isometrically rigid, relying on the quasi-isometric invariance of strong accessibility for word hyperbolic groups.

1. Introduction

A *Kleinian group* G is a discrete subgroup of $\mathbb{P}SL_2(\mathbb{C})$. It acts properly discontinuously on the hyperbolic 3-space $\mathbb{H}^3$ via orientation-preserving isometries and it acts on the Riemann sphere $\widehat{\mathbb{C}}$ via Möbius transformations. The latter action is usually not properly discontinuous: there is a canonical and invariant partition

$$\widehat{\mathbb{C}} = \Omega_G \sqcup \Lambda_G$$

where Ω_G denotes the *ordinary set*, which is the largest open set of $\widehat{\mathbb{C}}$ on which G acts properly discontinuously, and where Λ_G denotes the *limit set*, which is the minimal G-invariant compact subset of $\widehat{\mathbb{C}}$.

As Poincaré observed, we may identify the Riemann sphere with the boundary at infinity of the three-dimensional hyperbolic space [**Poi**]. Explicitly, let us consider the open unit ball in $\mathbb{R}^3$ as a model of $\mathbb{H}^3$ and the unit sphere $\mathbb{S}^2$ for the Riemann sphere. One obtains in this way an action of a Kleinian group G on the closed unit ball. With this identification in mind, the group G preserves the convex hull $\mathrm{Hull}(\Lambda_G)$ of its limit set in $\mathbb{H}^3$. The group G is *convex-cocompact* if its action is cocompact on $\mathrm{Hull}(\Lambda_G)$.

This class of groups plays an essential role in the classifcation of compact 3-manifolds. When G is torsion-free, we may associate a 3-manifold $M_G = (\mathbb{H}^3 \cup \Omega_G)/G$, canonically endowed with a complete hyperbolic structure in its interior, which is called the *Kleinian manifold* associated to G. If G is a torsion-free Kleinian group, then G is convex-cocompact if and only if M_G is compact. Note that M_G is also orientable, irreducible and its fundamental group is infinite (as soon as M_G

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is not a closed ball) and contains no subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Conversely, Perel'man proved that an orientable, irreducible compact 3-manifold with infinite fundamental group which contains no subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ is isomorphic to some M_G, where G is a convex-cocompact Kleinian group (when M is Haken, this was previously established by Thurston).

Characterizing this class of convex-cocompact Kleinian groups is a natural and interesting problem. From the point of view of geometric group theory, one tries to understand the properties of a group by studying the different actions it admits on metric spaces. To start with, a finitely generated group acts by left-translation on the Cayley graph X associated to any of its finite generating sets. If such a graph is equipped with the length metric which makes each edge isometric to the segment $[0, 1]$, then the action of G becomes *geometric*: the group G acts by isometries (the action is distance-preserving), properly discontinuously (for any compact subsets K and L of X, at most finitely many elements g of G will satisfy $g(K) \cap L \neq \emptyset$) and cocompactly (the orbit space X/G is compact).

The classification of finitely generated groups up to *quasi-isometry* is a central issue: a quasi-isometry between metric spaces X and Y is a map $\varphi : X \to Y$ such that there are constants $\lambda > 1$ and $c > 0$ such that:

- (quasi-isometric embedding) for all $x, x' \in X$, the two inequalities
$$\frac{1}{\lambda} d_X(x, x') - c \leq d_Y(\varphi(x), \varphi(x')) \leq \lambda d_X(x, y) + c$$
hold and
- the c-neighborhood of the image $f(X)$ covers Y.

This defines in fact an equivalence relation on metric spaces. Note that any two locally finite Cayley graphs of the same group are quasi-isometric; this fact enables us to consider the quasi-isometry class of a finitely generated group (through the class of its locally finite Cayley graphs). More generally, Švarc-Milnor's lemma asserts that there is only one geometric action of a group on a proper geodesic metric space up to quasi-isometry [**GdlH**, Prop. 3.19].

On the other hand, quasi-isometric groups may be very different one from another. For instance, it is not clear that a group quasi-isometric to a linear group is itself linear.

Our main result says that a group G is virtually a convex-cocompact Kleinian group if it looks like one:

Theorem 1.1 (quasi-isometric rigidity). *The class of convex-cocompact Kleinian groups is quasi-isometrically rigid. More precisely, a finitely generated group quasi-isometric to a convex-cocompact Kleinian group contains a finite index subgroup isomorphic to a (possibly different) convex-cocompact Kleinian group.*

This theorem was derived in [**Haï**] from an indirect approach, using rather heavy material. In the present paper, we would like to offer a more direct proof, avoiding many technical parts of the aformentioned paper, relying instead on three-dimensional topological methods when possible. We assume nonetheless some familiarity with Bass-Serre theory.

There are several cases which were already known prior to this work.

Let us assume that G is a finitely generated group quasi-isometric to a convex-cocompact Kleinian group K. The following cases are already known.

(1) If $\Lambda_K = \emptyset$ then K is finite, so G is finite as well, and we may consider the trivial group.
(2) If Λ_K consists of two points, then K contains an infinite cyclic subgroup of finite index, so G as well [**SW**], hence G contains a finite index subgroup isomorphic to the cyclic and Kleinian group $\langle z \mapsto 2z \rangle$.
(3) If Λ_K is a Cantor set, then K is virtually free, so G as well [**Sta, SW**] and there is a finite index free subgroup H which is isomorphic to the fundamental group of a handlebody. This implies that H is isomorphic to a convex-cocompact Kleinian group —a so-called Schottky group.
(4) If Λ_K is homeomorphic to the unit circle, then G contains a finite index subgroup isomorphic to a cocompact Fuchsian group according to [**CJ, Gab**].
(5) If $\Lambda_K = \widehat{\mathbb{C}}$ then G contains a finite index subgroup isomorphic to a cocompact Kleinian group [**CC**]; see also Corollary 3.6.
(6) If Λ_K is homeomorphic to the Sierpiński carpet, then G is commensurable to K [**Fri**]; this also follows straightfowardly from [**BKM**], see Corollary 3.10.

Our method of proof provides us with the following result.

THEOREM 1.2. *Let G be a word hyperbolic group and let us assume that its boundary is homeomorphic to the limit set of a convex-cocompact Kleinian group which contains no subset homeomorphic to the Sierpiński carpet. Then G contains a finite index subgroup isomorphic to a convex-cocompact Kleinian group.*

In our setting, an *elementary group* is a group which is either finite or virtually cyclic. We will assume that G is non-elementary with limit set different from a circle and a sphere, unless specifically stated.

Outline of the paper.— In the next section, we quickly review properties of 3-manifolds and their relationships with convex-cocompact Kleinian groups. Section 3 is devoted to word hyperbolic groups. After recalling their definition and main properties, we explain how the previously known cases of quasi-isometric rigidity may be established, using analytic properties of their boundaries. We conclude the section with a word on quasiconvex subgroups. Sections 4 and 5 are devoted to canonical splittings of word hyperbolic groups inspired by 3-manifolds. It is proved that strong accessibility is invariant under quasi-isometries (Theorem 5.2). The proof of the main theorem is given in Section 6. We introduce the notion of a regular JSJ-decomposition for groups which enables one to build manifolds and we adapt the notions of acylindrical pared manifolds to groups. Starting from a group G quasi-isometric to a convex-cocompact Kleinian group K, the proof consists in building a 3-manifold with fundamental group isomorphic to (a finite-index subgroup of) G and in applying Thurston's uniformization theorem. We proceed by induction on the hierarchy of G over elementary groups provided by K. The initial cases are essentially already known. For the inductive step, we split the group over elementary groups and apply the induction hypothesis to obtain a finite collection of pared manifolds that are to be glued together. The gluing is made possible thanks to the separability of quasiconvex subgroups provided by the theory of Wise, Agol et al. on word hyperbolic groups acting on CAT(0) cube complexes (Theorem 6.2); it enables us to replace G by a subgroup of finite index G' such that the splitting is sufficiently regular so that the pared submanifolds can be pieced together to yield a Haken compact manifold with fundamental group isomorphic to G'. We conclude the paper with an example given by Kapovich and Kleiner which shows that taking a finite index subgroup can be necessary. We also provide some examples showing that the residual limit set of Kleinian groups does not depend only on its isomorphism class: this justifies the efforts done in the inductive step.

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2. Hyperbolizable 3-manifolds

We quickly review definitions and properties of 3-manifolds and of convex-cocompact Kleinian groups. Basic references include [**Thu2, Mor, Mad**]. The following exposition is inspired by [**And**].

Recall that a Kleinian group K is convex-cocompact if $M_K = (\mathbb{H}^3 \cup \Omega_K)/K$ is compact. Ahlfors showed that if the limit set is not the whole Riemann sphere then it has measure 0. Furthermore, each connected component of its limit set is locally connected [**AnM**].

An orientable 3-manifold M is *hyperbolizable* if there exists a convex-cocompact Kleinian group such that M is homeomorphic to M_K (this whole presentation rules out tori in ∂M since they are not relevant to the present work). We say that M is *uniformized* by K. Note that K is isomorphic to the fundamental group of M, and that it is necessarily word hyperbolic if it contains no subgroup isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$, see the next section. Moreover, the boundary ∂M is a union of finitely many hyperbolic compact surfaces.

Let M be a compact hyperbolizable 3-manifold with boundary. A surface S is *properly embedded* in M if S is compact and orientable and if either $S \cap \partial M = \partial S$ or S is contained in ∂M. A *compression disk* D is an embedded disk such that $\partial D \subset \partial M$ and ∂D is homotopically non trivial in ∂M. A properly embedded surface S is *incompressible* if S is not homeomorphic to the 2-sphere and either S is a compression disk or if the inclusion $i : S \to M$ gives rise to an injective morphism $i_* : \pi_1(S, x) \to \pi_1(M, x)$. The *double* of a manifold M with boundary is the union of M and of a copy of itself glued along its boundary. A surface S in M is *boundary incompressible* if its double is incompressible in the double of M. A *Haken manifold* is a manifold which contains an incompressible surface. In our situation, as $\partial M \neq \emptyset$, M is always Haken as soon as it fundamental group is infinite.

We say that M has *incompressible boundary* if each component of ∂M is incompressible. This is equivalent to the connectedness of the limit set of the group uniformizing M.

A surface S in M is *non-peripheral* if it is properly embedded and if the inclusion $i : S \to M$ is not homotopic to a map $f : S \to M$ such that $f(S) \subset \partial M$. A surface S is *essential* if it is properly embedded, incompressible, boundary incompressible and non-peripheral. An *acylindrical compact manifold* has incompressible boundary and no essential annuli: the limit set of any uniformizing group is homeomorphic to the Sierpiński carpet.

A compact *pared manifold* (M, P) is given by a 3-manifold M as above together with a finite collection of pairwise disjoint incompressible annuli $P \subset \partial M$ such that any cylinder in $C \subset M$ with boundaries in P can be homotoped relatively to its boundary into P. We say the paring is *acylindrical* if $\partial M \setminus P$ is incompressible and every incompressible cylinder disjoint from P and with boundary curves in ∂M can be homotoped into $\partial M \setminus P$ relatively to ∂M.

If M has compressible boundary, it can be cut along *compression disks* into finitely many pieces each of which has incompressible boundary. Given a compact hyperbolizable manifold M with incompressible boundary, we may cut it into finitely many pieces along essential annuli so that the remaining pieces are acylindrical pared manifolds. Forgetting the paring, we may then iterate the above procedure by looking for compression disks and essential annuli.

It is a consequence of Haken's finiteness theorem that this procedure has to stop in finite time, yielding a finite number of compact manifolds each of which is either a ball or an acylindrical manifold. The collection of all the manifolds that are obtained from M defines a *hierarchy* of the manifold and of its fundamental group.

We will use the following form of Thurston's uniformization theorem for Haken manifolds:

THEOREM 2.1. *Let M be a compact irreducible orientable Haken 3-manifold with word hyperbolic fundamental group. Then M is hyperbolizable.*

3. Hyperbolicity

Background on word hyperbolic groups include [**Gro, GdlH, KB**].

Let X be a metric space. It is *geodesic* if any pair of points $\{x, y\}$ can be joined by a (geodesic) segment i.e, a map $\gamma : [0, d(x, y)] \to X$ such that $\gamma(0) = x$, $\gamma(d(x, y)) = y$ and $d(\gamma(s), \gamma(t)) = |t - s|$ for all $s, t \in [0, d(x, y)]$. The metric space X is *proper* if closed balls of finite radius are compact.

A *triangle* Δ in a metric space X is given by three points $\{x, y, z\}$ and three segments joining them two by two. Given a constant $\delta \geq 0$, the triangle Δ is δ-*thin* if any side of the triangle is contained in the δ-neighborhood of the two others.

DEFINITION 3.1 (Hyperbolic spaces and groups). *A geodesic metric space is hyperbolic if there exists $\delta \geq 0$ such that every triangle is δ-thin. A group G is word hyperbolic if it acts geometrically on a proper, geodesic hyperbolic metric space.*

Basic examples of hyperbolic spaces are the simply connected Riemannian manifolds $\mathbb{H}^n$ of sectional curvature (-1) and $\mathbb{R}$-trees. In particular since the action of a convex-compact Kleinian group G is cocompact on $\mathrm{Hull}\,\Lambda_G$, the group G is word hyperbolic.

Shadowing lemma.— A *quasigeodesic* is the image of an interval by a quasi-isometric embedding. The shadowing lemma asserts that, given δ, λ and c, there is a constant $H = H(\delta, \lambda, c)$ such that, for any (λ, c)-quasigeodesic q in a proper geodesic δ-hyperbolic metric space X, there is a geodesic γ at Hausdorff distance at most H.

It follows from the shadowing lemma that, among geodesic metric spaces, hyperbolicity is invariant under quasi-isometry : if X, Y are two quasi-isometric geodesic metric spaces, then X is hyperbolic if and only if Y is hyperbolic.

Compactification.— A proper geodesic hyperbolic space X admits a canonical compactification $X \sqcup \partial X$ at infinity. This compactification can be defined by looking at the set of rays i.e., isometric embeddings $r : \mathbb{R}_+ \to X$, up to bounded Hausdorff distance. The topology is induced by the uniform convergence on compact subsets. The boundary can be endowed with a family of *visual distances* d_v compatible with its topology i.e., which satisfy

$$d_v(a,b) \asymp e^{-\varepsilon d(w,(a,b))}$$

where $w \in X$ is any choice of a base point, $\varepsilon > 0$ is a *visual parameter* chosen small enough, and (a,b) is any geodesic asymptotic to rays defining a and b.

If $\Phi : X \to Y$ is a quasi-isometry between two proper geodesic hyperbolic spaces, then the shadowing lemma implies that Φ induces a homeomorphism $\phi : \partial X \to \partial Y$. This means that a word hyperbolic group G admits a topological boundary ∂G defined by considering the boundary of any proper geodesic metric space on which G acts geometrically.

In the case of a convex-compact Kleinian group K, a model for the boundary ∂K is given by its limit set Λ_K.

3.1. Analytic aspects. A general principle asserts that a word hyperbolic group is determined by its boundary. More precisely, Paulin proved that the quasi-isometry class of a word hyperbolic group is determined by its boundary equipped with its quasiconformal structure [**Pau**]. Let us recall some definitions. A homeomorphism $h : X \to Y$ between metric spaces is called *quasi-Möbius* [**Väi**] if there exists a homeomorphism $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any distinct points $x_1, x_2, x_3, x_4 \in X$,

$$[h(x_1) : h(x_2) : h(x_3) : h(x_4)] \le \theta([x_1 : x_2 : x_3 : x_4])$$

where

$$[x_1 : x_2 : x_3 : x_4] = \frac{|x_1 - x_2| \cdot |x_3 - x_4|}{|x_1 - x_3| \cdot |x_2 - x_4|}\,.$$

Quasi-Möbius maps are stable under composition. The boundary ∂G of a word hyperbolic group G is endowed with a *conformal gauge* i.e., a family of metrics which are pairwise quasi-Möbius equivalent. These metrics are exactly those metrics compatible with the topology of ∂G for which the action of G is *uniformly quasi-Möbius*, meaning that the distortion control θ is independent of $g \in G$. Visual distances are examples of metrics of the gauge.

Quasi-isometries provide natural examples of quasi-Möbius maps as well:

Theorem 3.2. *A (λ, c)-quasi-isometry between proper, geodesic, hyperbolic metric spaces extends as a θ-quasi-Möbius map between their boundaries, where θ only depends on λ, c, the hyperbolicity constants and the visual parameters.*

This result takes its roots in the work of Efremovich and Tihomirova [**ET**]; see also [**Mag**] where quasi-isometries are explicitly defined and where Theorem 3.2 is proved for real hyperbolic spaces. The key fact of its proof is that the crossratio $[x_1 : x_2 : x_3 : x_4]$ is measured by the distance between geodesics (x_1, x_2) and (x_3, x_4), a quantity quasi-preserved by quasi-isometries, cf. [**Pau**, Prop. 4.5] for a proof in the present setting.

Paulin's result reads

Theorem 3.3 (Paulin [**Pau**]). *Two non-elementary word hyperbolic groups are quasi-isometric if and only if there is quasi-Möbius homeomorphism between their boundaries.*

3.2. Convergence actions. If G is a word hyperbolic group acting geometrically on a proper hyperbolic metric space X, then its action extends as a uniformly quasi-Möbius action on ∂X. The action of G is *a convergence action* i.e., its diagonal action on the set of distinct triples is properly discontinuous, and is moreover *uniform* i.e., its action is also cocompact on the set of distinct triples. As for Kleinian groups, the *limit set* Λ_G is by definition the unique minimal closed invariant subset of X. It is empty if G is finite. These properties characterize word hyperbolic groups and their boundaries:

Theorem 3.4 (Bowditch [**Bow2**]). *Let G be a convergence group acting on a perfect metrizable space X. The action of G is uniform on Λ_G if and only if G is word hyperbolic and there exists an equivariant homeomorphism between Λ_G and the boundary at infinity ∂G of G.*

In our setting, we start with a finitely generated group G quasi-isometric to a convex-cocompact Kleinian group K. Since K is convex-cocompact, K is word hyperbolic, hence G as well. Thus, Theorem 3.4 implies that ∂G is homeomorphic to Λ_K. Moreover, we obtain in this way a quasi-Möbius action of the group G on $\Lambda_K \subset \widehat{\mathbb{C}}$. Note that the action of G need not be related to the action of K. Let $F < G$ be the kernel of the action of G on ∂G and $G' = G/F$. Since the action of G is properly discontinuous, F is finite and G' is quasi-isometric to G, hence $\partial G'$ is canonically identified with ∂G.

3.3. Quasi-Möbius groups versus Möbius groups. We recall some results which justify the quasi-isometric rigidity of some of the cases listed in the introduction. Let us begin with a general result which served as a motivation for the whole work.

Theorem 3.5 (Sullivan, [**Sul**]). *A countable group of uniformly quasi-Möbius transformations on $\widehat{\mathbb{C}}$ is conjugate to a group of Möbius transformations.*

Corollary 3.6 (Cannon and Cooper, [**CC**]). *If G is quasi-isometric to $\mathbb{H}^3$, then G contains a finite index subgroup isomorphic to a cocompact Kleinian group.*

Proof. According to the previous discussion, G acts on $\widehat{\mathbb{C}}$ as a uniform quasi-Möbius group. Hence Theorem 3.5 implies that the action of G is conjugate to a group of Möbius transformations. This action extends as an action by isometries on $\mathbb{H}^3$. Since the action is a uniform convergence action, we may check as follows that G is a cocompact Kleinian group: to any triple on $\widehat{\mathbb{C}}$, we associate the center in $\mathbb{H}^3$ of the ideal triangle with those points as vertices. We may check that since the action of G is properly discontinuous and cocompact on the set of triples, it is also properly discontinuous and cocompact on $\mathbb{H}^3$.

If the action of G on its boundary is faithful, then the corollary is proved. Otherwise, we have just proved that it admits a quotient $G' = G/F$ by a finite subgroup F isomorphic to a cocompact Kleinian group. To prove that G itself is virtually isomorphic to a convex-cocompact Kleinian group, we need to invoke the theory of word hyperbolic groups acting on CAT(0) cube complexes that we state as Lemma 3.7 below. Therefore, we may find $G'' < G$ of finite index such that $G'' \cap F = \{e\}$ implying that its action is faithful. □

LEMMA 3.7. *Let G be a finitely generated group which contains a normal finite subgroup F such that G/F is isomorphic to a convex-cocompact Kleinian group. Then G is residually finite.*

A proof goes as follows: according to [**BW**], the group G/F acts geometrically on a CAT(0) cube complex, so G as well. Hence G contains a finite index subgroup which acts specially in the sense of Haglund and Wise [**HW**] according to Agol [**Ago**]. Therefore, G is residually finite [**HW**].

We now recall the rigidity of groups acting on the circle.

THEOREM 3.8 (Hinkkanen, Markovic, [**Hin1, Hin2, Mak**]). *Any uniformly quasi-Möbius group of homeomorphisms on the unit circle is quasisymmetrically conjugate to a group of Möbius transformations.*

Let us now pass to groups acting on Sierpiński carpets.

THEOREM 3.9 (Bonk, Merenkov, Kleiner, [**BKM**]). *Any quasi-Möbius selfmap of a round Sierpiński carpet of measure zero is the restriction of a Möbius transformation.*

COROLLARY 3.10 (Frigerio, [**Fri**]). *If G is quasi-isometric to the fundamental group K of a compact hyperbolic 3-manifold with totally geodesic boundary, then G is commensurable to K.*

PROOF. Since K is the fundamental group of a compact hyperbolic 3-manifold with totally geodesic boundary, its boundary is a measure zero round Sierpiński carpet. Let G_M denote the set of quasi-Möbius selfhomeomorphisms of Λ_K. Note that G_M contains $K \cup G'$, where G' consists of the largest quotient which has a faithful action on its boundary as above.

According to Theorem 3.9, the action of G_M extends to an action of Möbius transformations on $\widehat{\mathbb{C}}$. It is clearly discrete since any sequence which tends uniformly to the identity will have to eventually stabilize at least three circles, implying that such a sequence is eventually the identity. Therefore, G_M is a convergence group, and since it contains K, its action is uniform on Λ_K, hence it is word hyperbolic. Since the boundaries of K and G_M coincide, K has finite index in G_M [**KS**]. The same holds for G'.

We conclude as above: if $G = G'$, then the corollary is proved. Otherwise, we apply Lemma 3.7 to conclude that G is residually finite. Therefore, we may find $G'' < G$ of finite index such that $G'' \cap F = \{e\}$ implying that its action is faithful. □

Thus, we assume throughout this paper that G is none of the above classes.

3.4. Quasiconvex subgroups. Basic references include [**KS**]. Let X be a proper, geodesic, hyperbolic metric space. A K-*quasiconvex* subset $Y \subset X$ has the property that any geodesic segment joining two points of Y remains in the K-neighborhood of Y. Note that quasiconvexity is a property invariant under quasi-isometries. A subgroup H of a hyperbolic group G is *quasiconvex* if H is quasiconvex in any locally finite Cayley graph of G.

If a group G acts properly discontinuously by isometries on a proper geodesic hyperbolic metric space, its *limit set* Λ_G is defined as the set of accumulation points at infinity of any orbit $G(x)$, $x \in X$. This set is G-invariant.

Proposition 3.11. *Let $\Phi : G_1 \to G_2$ be a quasi-isometry, $\phi : \partial G_1 \to \partial G_2$ its boundary map, and let $H_j \subset G_j$ be quasiconvex subgroups. If $\phi(\Lambda_{H_1}) = \Lambda_{H_2}$ then $\Phi(H_1)$ is at bounded distance from H_2 so H_1 and H_2 are quasi-isometric.*

Proof. If H_1 is finite, then Λ_{H_1} is empty, so Λ_{H_2} is also empty and H_2 is finite as well. Otherwise, it follows from [**KS**, Prop. 3.4] that H_j is at bounded distance from the convex hull $\mathcal{C}(\Lambda_{H_j})$, $j = 1, 2$, consisting of the union of geodesics with endpoints in the respective limit sets. By the shadowing lemma, $\Phi(\mathcal{C}(\Lambda_{H_1}))$ is quasiconvex, at bounded distance from $\mathcal{C}(\Lambda_{H_2})$, implying the proposition. □

Another proof could have been obtained from Theorem 3.3.

4. Canonical splittings

We describe well-known splittings of word hyperbolic groups which correspond to splitting a manifold along compression disks and essential annuli. In each case, the Bass-Serre theory will provide us with a simplicial action of G on a simplicial tree T with no edge inversions such that T/G is finite and all vertex (stabilizers) groups and edge (stabilizers) groups are quasiconvex. Note that it is enough to check that the edge groups are quasiconvex to ensure the vertex groups are as well [**Bow1**, Prop. 1.2]. We refer to [**Ser, SW**] for a general introduction to the Bass-Serre theory.

In both cases, the splitting will be maximal in a precise sense.

4.1. Splittings over finite groups. Let G be a non-elementary word hyperbolic group. If it is not one-ended then it splits over a finite group [**Sta**]. By [**Dun**], there is a maximal splitting over finite groups such that each vertex group is finite or one-ended; when G is torsion-free, it leads to a free product of a free group with finitely many one-ended groups.

More precisely, the group G acts simplicially on a simplicial tree T with no edge inversions such that T/G is finite and

- all edge groups are finite;
- all vertex groups are finite or one-ended;

Moreover, the one-ended vertex groups stabilise a nontrivial connected component of the boundary. Conversely, any nontrivial component of ∂G is stabilised by a one-ended vertex group.

We call this splitting a *DS-splitting*.

Proposition 4.1. *Let G and H be two quasi-isometric word hyperbolic groups. Then each infinite vertex group of the DS-splitting of G is quasi-isometric to an infinite vertex group of the DS-splitting of H, and vice-versa.*

This proposition holds very generally for finitely presented groups [**PW**]. We provide an elementary proof in our setting.

Proof. Let $\Phi : G \to H$ be a quasi-isometry. According to Theorem 3.2, it extends as a quasi-Möbius homeomorphism $\phi : \partial X \to \partial Y$. In particular, ϕ maps connected components of ∂X to components of ∂Y. Therefore, if G_v is a one-ended vertex of G in its DS-splitting with limit set Λ_v, then $\phi(\Lambda_v)$ is a component of ∂H, hence is stabilised by a one-ended vertex group H_v. Proposition 3.11 implies that G_v and H_v are quasi-isometric. □

4.2. Splittings over 2-ended groups. We summarize briefly the JSJ decomposition of a non-Fuchsian one–ended word hyperbolic group G following Bowditch [**Bow1**].

There exists a canonical simplicial minimal action of G on a simplicial tree $T = (V, E)$ without edge inversions such that T/G is a finite graph and which enjoys the following properties, cf. [**Bow1**, Thm 5.28, Prop. 5.29]. If v is a vertex (resp. e an edge), we will denote by G_v (resp. G_e) its stabilizer, and by Λ_v (resp. Λ_e) the limit set of G_v (resp. G_e). Let E_v denote the set of edges incident to $v \in T$. Every vertex and edge group is quasiconvex in G. Each edge group G_e is two-ended and $\partial G \setminus \Lambda_e$ is not connected. A vertex v of T belongs to exactly one of the following three exclusive types.

> **Type I (elementary).**— The vertex has bounded valence in T. Its stabilizer G_v is two-ended, and the connected components of $\partial G \setminus \Lambda_v$ are in bijection with the edges incident to v.
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> **Type II (surface).**— The limit set Λ_v is cyclically separating and the stabilizer G_v of such a vertex v is a non-elementary virtually free group canonically isomorphic to a convex-cocompact Fuchsian group. The incident edges are in bijection with the peripheral subgroups of that Fuchsian group.
>
> **Type III (rigid).**— Such a vertex v does not belong to any class above. Its stabilizer G_v is a non-elementary quasiconvex subgroup. Every local cut point of ∂G in Λ_v is in the limit set of an edge stabilizer incident to v and Λ_v is a maximal closed subset of ∂G with the property that it cannot be separated by two points of ∂G.

By construction, no two vertices of the same type are adjacent, nor surface type and rigid can be adjacent either. This implies that every edge contains a vertex of elementary type. The action of G preserves the types. Therefore, the edges incident to a vertex v of surface type or rigid type are split into finitely many G_v-orbits.

PROPOSITION 4.2. *Let G and H be two quasi-isometric word hyperbolic groups. Then each vertex and edge group of the JSJ-splitting of G is quasi-isometric to a vertex and edge group of the JSJ-splitting of H respectively, and vice-versa.*

PROOF. We proceed as in Proposition 4.1. Let $\Phi : G \to H$ be a quasi-isometry. According to Theorem 3.2, it extends as a quasi-Möbius homeomorphism $\phi : \partial X \to \partial Y$. In particular, ϕ maps the local cut points of ∂X to the local cut points of ∂Y. Therefore, since the JSJ decomposition is built from the local cut points of the boundaries of the groups, ϕ will map limit sets of edge groups to limit sets of edge groups, limit sets of vertex groups to limit sets of vertex groups of the same type. Thus, if G_v is a vertex group of G in its JSJ-splitting with limit set Λ_v, then $\phi(\Lambda_v)$ is the limit set of a vertex group H_v in the JSJ-decomposition of H. Proposition 3.11 implies that G_v and H_v are quasi-isometric. □

5. Strong accessibility

What follows is inspired by similar properties of compact 3-manifolds, cf. § 2. We first show how to combine the DS- and JSJ- splittings together, following Drutu and Kapovich [**DK**].

Proposition 5.1. *Let G be a word hyperbolic group. If G is not quasi-isometric to a Fuchsian group, then G acts on a simplicial tree T with no edge inversions such that*

- *the quotient T/G is finite;*
- *the edge stabilizers are elementary groups;*
- *there is an equivariant projection $p : T \to T_{DS}$ such that $p^{-1}(v)$ provides the JSJ-decomposition of G_v, for each non-Fuchsian one-ended vertex.*

Moreover, this splitting is maximal with respect to elementary groups.

Proof. Let us consider the DS-splitting of G: it is given by a graph of groups $\mathcal{G}_{DS} = (\Gamma_{DS}, \{G_v\}, \{G_e\}, G_e \hookrightarrow G_{t(e)})$, whose fundamental group is isomorphic to G and where the edge groups are all finite and the vertex groups are either finite or one-ended. Let G_v be a non-Fuchsian one-ended vertex group, and let us consider its JSJ-decomposition given by a graph of groups $\mathcal{G}_v = (\Gamma_v, \{H_s\}, \{H_a\}, H_a \hookrightarrow H_{t(a)})$. We wish to refine $\mathcal{G}_{DS}$ and insert $\mathcal{G}_v$.

Let e be an incident edge to v in Γ_{DS}. Note that G_e is a subgroup of G_v hence acts on the Bass-Serre tree T_v associated to $\mathcal{G}_v$. Since G_e is finite, it admits a fixed vertex point $s' \in T_v$. Its stabilizer $H_{s'}$ is conjugate to a vertex group H_s of $\mathcal{G}_v$ by an element $h_{s'} \in G_v$.

We may now define a new graph of groups $\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\}, G_e \hookrightarrow G_{t(e)})$ as follows. We let Γ be the union $(\Gamma \setminus \{v\}) \cup \Gamma_v$ that we tie together by attaching each incident e in Γ_{DS} to the vertex $s \in \Gamma_v$ defined above. One may check that the natural embeddings yield a graph of groups whose fundamental group is isomorphic to G; see [**DK**, § 18.6] for details.

Proceeding similarly for each non-Fuchsian one-ended vertex group in Γ_{DS} provides us with the desired splitting. □

Let G be a word hyperbolic group. Applying Proposition 5.1, we let $\mathcal{H}_1$ denote all the vertex groups in the quotient T/G and $\mathcal{H}_1' \subset \mathcal{H}_1$ be those which are non-Fuchsian.

We proceed inductively so that $\mathcal{H}_{n+1}$ consists of the set of isomorphic classes of vertex groups appearing in the decomposition of vertex groups in $\mathcal{H}_n'$ given by Proposition 5.1 and let $\mathcal{H}_{n+1}' \subset \mathcal{H}_{n+1}$ be those classes of groups that are non-Fuchsian.

A group G is *strongly accessible* if the process ends i.e., $\mathcal{H}_n'$ consists of one-ended groups which are either Fuchsian or such that their boundary have no local cut points. The collection of conjugacy classes $\cup\mathcal{H}_n$ defines a *hierarchy* of the group G and the smallest such ℓ with the property that $\mathcal{H}_\ell$ consists of groups which cannot be split further is the *depth* of the hierarchy.

If G has no element of order two, then this process stops in finite time [**DP, LT**], exactly as for Haken manifolds which are cut into finitely many balls. We will not need these results here, but rely instead on the following quasi-isometric invariance property, which may be of independent interest.

Theorem 5.2 (QI-invariance of strong accessibility). *Let G_1 and G_2 be two quasi-isometric word hyperbolic groups. Then G_1 and G_2 are either both strongly accessible or not. In the former case, the depth of their hierarchy coincide and, for all $n \geq 1$, each group in $\mathcal{H}_n(G_1)$ is quasi-isometric to a group in $\mathcal{H}_n(G_2)$, and vice-versa.*

Proof. Let us proceed by induction on the depth. Let us assume that G_1 is either virtually free, virtually Fuchsian, or with no local cut points. Then the same holds for G_2 since the boundaries are homeomorphic. Let us now assume that G_1 has a hierarchy of depth $\ell \geq 1$. Let us apply Proposition 5.1 to G_1 and G_2. Note that Proposition 4.1 implies that G_1 and G_2 have the same vertex groups up to quasi-isometry in their respective DS-splittings. Moreover, Proposition 4.2 implies that the vertex groups appearing in the JSJ splittings from the above vertex groups are also the same up to quasi-isometry. Therefore, groups in $\mathcal{H}_1(G_1)$ are quasi-isometric to groups in $\mathcal{H}_1(G_2)$ and vice-versa. Furthermore, elements of $\mathcal{H}_1(G_1)$ have depth at most $\ell - 1$, so we may apply the induction hypothesis. □

6. Quasi-isometric rigidity

Let G be a finitely generated group quasi-isometric to a convex-cocompact Kleinian group. The basic idea is to build a compact 3-manifold with fundamental group isomorphic to (a finite-index subgroup of) G and then to apply Thurston's uniformization theorem. This construction will proceed by induction on the hierarchy of G, and will require to have a *regular* JSJ decomposition as defined below. In general, a given group does not admit such a decomposition. We will show that we may obtain this property for a suitable subgroup of finite-index. Its existence will be granted by the separability of its quasiconvex subgroups (Theorem 6.2).

6.1. Regular JSJ decomposition. Let $M = M_K$ be a hyperbolizable manifold with incompressible boundary so that Λ_K is connected. Let us assume that M is built from gluing several manifolds $M_1, \ldots, M_k$ to a solid torus T along annuli whose cores are homologous to multiples of the core of T. In this case, these curves have to be all parallel in ∂T. By taking a suitable finite covering $M' = M_{K'}$ of M, we may then assume that each one of them generate the fundamental group of the torus. Thus, the fundamental group of such a lifted annulus A is generated by a primitive element g of K'. Its limit set Λ_g splits Λ_K into finitely many components, which are fixed under the action of g. This leads us to the following notion:

Definition 6.1 (Regular JSJ decomposition). *Let G be a one-ended word hyperbolic group and let us consider its JSJ decomposition. We say it is regular if the following properties hold:*

- *G is torsion free;*
- *every two-ended group H which appears as a vertex group is isomorphic to $\mathbb{Z}$ and stabilizes the components of $\partial G \setminus \Lambda_H$.*

Let us assume that the decomposition of a group G is regular and let G_v be a vertex group of elementary type G_v; since its action fixes the components of $\partial G \setminus \Lambda_v$, it is generated by a primitive element.

6.2. Separability properties. A subgroup H is *separable* in its ambient group G if, for any $g \in G \setminus H$, there is a finite index subgroup G' in G which contains H but not g.

Theorem 6.2. *Let G be a finitely generated group quasi-isometric to a convex-cocompact Kleinian group. Then its quasiconvex subgroups are separable.*

This theorem uses the recent breakthroughs of Wise, Agol, and their collaborators on word hyperbolic groups acting on CAT(0) cube complexes [**Wis, Ago**]. We only sketch the proof.

Proof. We use the QI-invariance of the strong accessibility together with Corollary 3.10 to prove that G admits a so-called virtual quasiconvex hierarchy. This implies according to Wise that the group is a so-called virtual special group, hence its quasiconvex subgroups are separable according to [**HW**]. See [**Haï**, § 8] for details. □

Separability is used to "clean" our group as explained by the next proposition and its corollaries. See [**Haï**, Prop. 7.2] for a proof.

Proposition 6.3. *Let $A' < A < G$ be groups with $[A : A'] < \infty$ and A' separable in G. Then there exist subgroups A'' and H with the following properties:*

(1) *H is a normal subgroup of finite index in G;*
(2) *$A'' = H \cap A'$ is a normal subgroup of finite index in A ;*
(3) *for all $g \in G$, $(gAg^{-1}) \cap H = gA''g^{-1}$.*

As a first application, we obtain from the finiteness of the conjugacy classes of torsion elements:

Corollary 6.4. *A residually finite word hyperbolic group is virtually torsion-free.*

Another simple application of Proposition 6.3 is the following reduction consequence.

Corollary 6.5. *Let G be the fundamental group of a finite graph of groups $\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\})$, where each edge group is quasiconvex in G. Let $A_v < G_v$ be subgroups of finite index. If the quasiconvex subgroups of G are separable, then G contains a normal finite index subgroup G' such that, for any $g \in G$, $gG_vg^{-1} \cap G'$ is a finite index subgroup of gA_vg^{-1}.*

Proof. It follows from [**Bow1**, Prop. 1.2] that the vertex groups are quasiconvex as well. We apply Proposition 6.3 to each triple (A_v, G_v, G) to obtain finitely many finite index subgroups G'_v of G and we let G' denote their intersection. The conclusion follows. □

We now derive the virtual regularity of the JSJ decomposition.

Corollary 6.6. *Let G be a finitely generated group quasi-isometric to a convex-cocompact Kleinian group. Then G contains a finite index subgroup such that its one-ended vertex groups in the DS-splitting have a regular JSJ-decomposition.*

Proof. It follows from Theorem 6.2 that every elementary group is separable. In particular, G contains a finite index subgroup H' which is torsion-free by Corollary 6.4. It follows that its action on ∂G is faithful. Let K' be one of the factors of the DS-splitting of H' which is one-ended but not Fuchsian.

Let us consider a set of representatives of elementary vertex groups $\{K'_v\}$ arising from its JSJ decomposition. Each one of them contains a cyclic subgroup A_v of finite index which stabilizes all the components of $\partial K \setminus \Lambda_v$. Corollary 6.5 applied to (A_v, G_v) produces a normal subgroup H of finite index in G. By construction, the vertices appearing in the DS-splitting of H admit a regular JSJ-decomposition; see [**Haï**, Thm 7.1] for details. □

6.3. Acylindrical pared groups. We adapt the notion of acylindrical pared 3-manifolds to groups. This point of view was first developed by Otal for free groups in [**Ota**]. Let G be a nonelementary word hyperbolic group. A *paring* will be given by a finite malnormal collection $P_G = \{C_1, \ldots, C_k\}$ of maximal 2-ended groups i.e., if $C_i \cap (gC_jg^{-1})$ is infinite for some $i, j \in \{1, \ldots, k\}$ and $g \in G$, then $i = j$ and $g \in C_i$ and such that G does not split over them. We will say that the paring is *acylindrical* if, whenever G splits over an elementary group H, the paring P_G does not split to define a paring of vertex groups — up to conjugacy.

FACT 6.7. *Let G be a non-Fuchsian one-ended word hyperbolic group with a regular JSJ decomposition. Let v be a vertex from its JSJ decomposition of rigid type and E_v denote the edges incident to v. Then $(G_v, \{G_e, e \in E_v/G_v\})$ is an acylindrical pared group.*

This fact follows from the maximality of the decomposition.

Note that an acylindrical pared compact hyperbolizable 3-manifold (M, P_M) gives rise to a canonical acylindrical pared group (K, P_K) by letting $K = \pi_1(M)$ and P_K denote the fundamental groups generated by the core of the annuli composing P_M. A quasi-isometry between two acylindrical pared groups (G, P_G) and (H, P_H) will be given by a quasi-isometry $\Phi : G \to H$ such that, for any $C_j \in P_G$, $\Phi(C_j)$ is at bounded distance from a conjugate of an element from P_H, and if any element of P_H is at bounded distance from the image of a conjugate of P_G. If G' is a finite index subgroup of an acylindrical pared group (G, P_G), then we associate an acylindrical paring to G' by considering a finite set of representative classes of $\{G' \cap gCg^{-1},\ g \in G, C \in P_G\}$. We will say that an acylindrical pared manifold (M, P_M) is quasi-isometric to an acylindrical pared group (G, P_G) if there is a quasi-isometry between its canonical acylindrical pared group (K, P_K) and (G, P_G).

Let us say that an acylindrical pared group (G, P_G) is *geometric* if there is a compact hyperbolic acylindrical pared 3-manifold (M, P) such that its canonical acylindrical pared group is isomorphic to (G, P_G) with $G = \pi_1(M)$.

Theorem 1.1 will follow from the following variant:

THEOREM 6.8 (pared quasi-isometric rigidity). *An acylindrical pared group (G, P_G) is virtually geometric if it is quasi-isometric to a compact hyperbolic acylindrical pared 3-manifold (M, P).*

The proof will be established by induction on the depth of the hierarchy of the group G.

The case of an acylindrical pared free group has been established by Otal [**Ota**]. Let us recall his theorem. Let (G, P) be an acylindrical pared word hyperbolic group. Define on ∂G an equivalence relation $\sim_P$ as follows: let $x \sim_P y$ if, either $x = y$ or if there is a subgroup $C \in P$, an element $g \in G$ such that $\{x, y\} = g(\Lambda_C)$. Set $\partial_P G = \partial G / \sim_P$ to be the quotient of ∂G by this relation $\sim_P$. This is always Hausdorff compact, connected and locally connected set.

THEOREM 6.9 (Otal). *An acylindrical pared free group $(\mathbb{F}, P)$ is geometric if $\partial_P \mathbb{F}$ is planar.*

COROLLARY 6.10. *An acylindrical pared free group $(\mathbb{F}, P)$ quasi-isometric to an acylindrical pared handlebody (M, P_M) is geometric.*

PROOF. Let (K, P_K) be the canonical acylindrical pared group associated to (M, P). According to [**Ota**, Proposition 0], $\partial_{P_K} K$ is planar, hence $\partial_P \mathbb{F}$ as well since the quasi-isometry $\Phi : (\mathbb{F}, P) \to (K, P_K)$ defines a homeomorphism $\phi : \partial\mathbb{F} \to \Lambda_K$ which maps the $\sim_P$-classes to $\sim_{P_K}$-classes. Therefore, Theorem 6.9 applies. □

We may now prove the main theorem.

PROOF. (Theorems 1.1 and 6.8) We first reduce the proof of Theorem 1.1 to the proof of Theorem 6.8. Let G be a group quasi–isometric to a convex-cocompact Kleinian group K. We might as well assume that G does not fall into one of the known cases mentioned in the introduction. Let us consider the splitting provided by Proposition 5.1 and its corresponding action G on T. Propositions 4.1 and 4.2 imply that each vertex group is quasi-isometric to a convex-cocompact Kleinian group. By Fact 6.7, the rigid vertices are canonically pared and quasi-isometric to an acylindrical pared manifold. Assuming for the time being Theorem 6.8, we may even conclude that the rigid vertices are virtually geometric.

Theorem 6.2, Corollaries 6.6 and 6.5 enable us to assume that, up to a finite index subgroup, G admits a regular JSJ decomposition with rigid vertex pared groups $(G_v, \{G_e, e \in E_v/G_v\})$ isomorphic to acylindrical pared manifolds (M_v, P_v). If v is of elementary type, then we associate a solid torus with parallel pairwise disjoint annuli on its boundary parametrized by the edges incident to v. If v is of surface type, then Theorem 3.8 implies that G_v is canonically isomorphic to a Fuchsian group F_v such that $S_v = \mathbb{D}/F_v$ is a surface with holes parametrized by E_v/G_v; we then consider $M_v = S_v \times [0, 1]$ with annuli A_e corresponding to the I-fibers of the holes. The finite graph T/G provides us with a manual to build a 3-manifold M with fundamental group isomorphic to G. We first glue the M_v's along annuli in order to obtain the vertices of the DS-decomposition, and then glue the resulting manifolds along disks [**SW**]. Thurston's hyperbolization theorem for Haken manifolds shows that M is hyperbolizable. This ends the proof of Theorem 1.1.

We now pass to the proof of Theorem 6.8. Let (G, P_G) be a group quasi–isometric to a pared manifold (M, P) and let K be a convex-cocompact Kleinian group representing its fundamental group. Since K is pared, its limit set is a proper subset of the sphere. According to Corollary 6.4 we may assume that G is torsion-free. Theorem 5.2 implies that G is strongly accessible so we may proceed by induction on the depth of its hierarchy.

If the depth is zero, then G is either

- free, to which we associate a handlebody; or
- Fuchsian, to which we associate an I-bundle by Theorem 3.8; or
- a carpet group, to which we associate an acylindrical manifold by Corollary 3.10.

We easily obtain an acylindrical pared manifold when G is free (Corollary 6.10), Fuchsian or a carpet group.

Let G be of depth $n + 1$, with $n \geq 0$, and let us assume that the theorem is established up to rank n. As for Theorem 1.1, the induction hypothesis enables us to assume that G admits a regular JSJ-decomposition and that each vertex group is the fundamental group of a pared manifold (N_v, Q_v).

The graph of groups coming from Proposition 5.1 provides us with a manual to build a 3-manifold N with fundamental group isomorphic to G by gluing the

N_v's along disks or annuli [**SW**]. In general, there are several choices for this last step. We need to make a suitable choice to make sure that N will admit a paring associated to P_G, cf § 7.1 for justifying the necessity of this step. The choice of the gluings will be made according to the embedding of ∂G in $\widehat{\mathbb{C}}$ provided by its quasi-isometry with K.

Let us explain how we use K to construct N. It follows from [**Ohs**] that there exists a Kleinian group K' isomorphic to K such that each element of P_K has become parabolic and its limit set is homeomorphic to $\partial_{P_K} K$. The canonical projection $\Lambda_K \to \partial_{P_K} K$ gives rise to a monotone map $p : (\widehat{\mathbb{C}}, \Lambda_K) \to (\widehat{\mathbb{C}}, \Lambda_{K'})$.

The quasi-isometry $\Phi : (G, P_G) \to (K, P_K)$ yields a homeomorphism $\phi : \partial G \to \Lambda_K$ which realizes a correspondence between the limit sets of gCg^{-1}, where $g \in G$ and $C \in P_G$, with the limit sets of kDk^{-1}, where $k \in K$ and $D \in P_K$. Thus, we obtain a homeomorphism $\varphi : \partial_{P_G} G \to \Lambda_{K'}$ which enables us to make G act by homeomorphisms on $\Lambda_{K'}$.

Since (K, P_K) is acylindrical, it follows that $M_{K'}$ has no essential annulus, so the closure of no component of $\Omega_{K'}$ separates $\Lambda_{K'}$, cf. [**Thu1**, Theorem 2]. Therefore, the action of G on $\Lambda_{K'}$ maps the boundaries of the components of $\Omega_{K'}$ to the boundaries of (possibly other) components of $\Omega_{K'}$. It follows that the action of G on Λ_K extends to a convergence action on $\widehat{\mathbb{C}}$ of G, cf. [**Haï**, Prop. 6.2].

This action enables us to realize geometrically the splitting of G given by Proposition 5.1: by [**MS**, Theorem 3.2], one may first find finitely many simple closed curves on Ω_K/G whose lifts form a collection Γ_1 of pairwise disjoint Jordan curves on $\widehat{\mathbb{C}}$; see also [**AbM**]. These curves realize the DS-splitting of G. Furthermore, the JSJ splitting of the G-stabilizers of non-trivial components of Λ_K different from circles can also be realised on $\widehat{\mathbb{C}}$ by considering all the hyperbolic geodesics in Ω_K joining the fixed points of the edge groups; this defines a second family Γ_2 of curves. Let $\Gamma = \Gamma_1 \cup \Gamma_2$: we define a tree T by considering

(a) the vertices to be the components of $\widehat{\mathbb{C}} \setminus \cup_{\gamma \in \Gamma} \overline{\gamma}$; to which we need to add the elementary vertices of the JSJ splittings; the latter are the fixed points of loxodromic elements which split a component into at least three components or which are at the intersection of two components corresponding to vertices of surface and rigid types;
(b) edges correspond to curves $\gamma \in \Gamma$ which are in the common boundary of the vertices, or on the boundary of one component and joining the fixed points of an elementary vertex.

We leave the reader check that G acts simplicially on T and that its action is isomorphic to the one given by Proposition 5.1.

We may now build a manifold N so that ∂N is homeomorphic to Ω_K/G. Let $v \in T$ be a vertex and $e \in E_v$ be an edge incident to v. This edge corresponds to a curve $\gamma \in \Gamma$ which defines a simple closed curve $\overline{\gamma} \subset \partial N_v$. Either it bounds a disk $D_e \subset \partial N_v$, or it is homotopically non-trivial and hence can be thickened to become an annulus $A_e \subset \partial N_v$. Now the graph of groups tells us how to glue the different vertex manifolds N_v's, $v \in T/G$, along the disks and annuli associated to the class of edges E_v/G_v. Let N be the manifold obtained by this construction. By construction, ∂N is homeomorphic to Ω_K/G.

It remains to check that P_G provides a paring of N. Let us first observe that the elements of P_G are maximal 2-ended subgroups of G, hence they are cyclic and generated by primitive elements. Note that P_K gives rise to a collection of pairwise disjoint curves in Ω_K joining pairs of fixed points of loxodromic elements. These points are also fixed points by the conjugates of P_G coming from the action of G on $(\widehat{\mathbb{C}}, \Lambda_K)$. Since ∂N is homeomorphic to Ω_K/G, this implies that P_G define curves on ∂N which are pairwise disjoint. So P_G defines a paring Q on N. The paring is acylindrical since this is the case for K.

Thurston's hyperbolization theorem for Haken manifolds shows that N is hyperbolizable. This proves that (G, P_G) is virtually geometric. □

6.4. Groups with homeomorphic boundaries. Since the proof of Theorem 1.2 is essentially the same as Theorem 1.1, we only sketch the argument. The main observation is that the hierarchy provided by Proposition 5.1 only depends on the topology of the limit set. Therefore, Theorem 5.2 still holds in this setting: if G is a word hyperbolic group with boundary homeomorphic to the one of a convex-cocompact Kleinian group, then G is strongly accessible since K is, the depth of their hierarchy coincide and, for all $n \geq 1$, the boundary of each group in $\mathcal{H}_n(G)$ is homeomorphic to the boundary of a group in $\mathcal{H}_n(K)$, and vice-versa. Assuming that K is furthermore torsion free and that Λ_K contains no subset homeomorphic to the Sierpiński case implies that the hierarchy's leaves contain only free groups, Fuchsian groups and elementary groups. Therefore, G will admit a quasiconvex hierarchy in the sense of Wise, so the conclusion of Theorem 6.2 and its corollaries will also hold in our context. Theorem 1.2 may be established by induction on the length of the hierarchy. The initial step consists in saying that a word hyperbolic whose boundary is either empty, consists of two points, a Cantor set or a circle is virtually isomorphic to a Fuchsian group: this was already known, cf. the introduction. The induction step proceeds as above: first construct a finite index subgroup G' in order to get regular JSJ decompositions of its maximal one-ended subgroups with geometric pared rigid subgroups, then apply the induction hypothesis to construct a compact 3-manifold with fundamental group isomorphic to G' and apply Thurston's uniformization theorem.

7. Examples

We provide examples explaining some of the difficulties that have to be dealt with.

7.1. Unstability of the residual limit set. In the proof of Theorem 6.8, we were very careful in the construction of the manifold M to ensure that P_G would be associated to a paring. This question is related to the residual limit set R_G of a Kleinian group, defined by Abikoff as those points of Λ_G which are not on the boundary of any component of Ω_G.

Let G be a finitely generated Kleinian group. According to Abikoff [**Abi**], the residual limit set has two different kinds of points: those which correspond to singleton connected components z of Λ_G for which there are components Λ_n, $n \geq 1$, of Λ_G, and components Ω_n of $\widehat{\mathbb{C}} \setminus \Lambda_n$ such that $(\Omega_n)_n$ defines a nested family of disks and $\cap\Omega_n = \{z\}$, and those points which belong to a non trivial connected component of Λ_G. In both cases, they are characterized by the fact that we may find a nested sequence of Jordan curves $\gamma_n \subset \Lambda_G$, components Ω_n of $\widehat{\mathbb{C}} \setminus \gamma_n$ such that $\cap\Omega_n$ is a residual point.

We give two examples that show that a homeomorphism between the limit sets of two isomorphic convex-cocompact Kleinian groups need not preserve the residual limit sets.

Let us denote by S_g a compact, closed, orientable surface of genus $g \geq 0$, and S_g^* the complement of an embedded closed disk in S_g.

7.1.1. *The residual set of an infinitely ended Kleinian group.* Let us first consider a hyperbolic compact 3-manifold M_1 with at least two non-homeomorphic boundary components S_1 and S_1'. Let $M_2 = S_g \times [0,1]$. Let us pick embedded closed disks $D_1 \subset S_1 (\subset \partial M_1)$, $D_1' \subset S_1' (\subset \partial M_1)$ and $D_2 \subset S_g \times \{0\} (\subset \partial M_2)$. Let M be the connected sum of M_1 and M_2 obtained by identifying D_1 and D_2 and M' by identifying D_1' with D_2. By van Kampen's theorem, both fundamental groups are isomorphic to the free product of the fundamental groups of both manifolds. Let K and K' be two convex-cocompact Kleinian groups uniformizing M and M' respectively.

Now, let us consider homotopically non-trivial curves $\gamma_1 \subset S_1 (\subset \partial M_1)$ and $\gamma_2 \subset S_g \times \{0\} (\subset \partial M_2)$. In M, their homotopy classes can be concatenated into a curve γ in ∂M. Its class defines a loxodromic element of K whose fixed points are non-residual, since it represents a boundary curve. On the other hand, in M', their concatenation defines a loop which has to go through the interior of M_1 to connect both pieces. Thus, any lift in its universal cover $\widetilde{M'}$ has to cross infinitely many lifts of M_1. This means that, in K', this curve represents a loxodromic element whose fixed points are in different nested intersections of lifts of S_1, hence in $R_{K'}$.

7.1.2. *The residual set of a one-ended Kleinian group.* We consider $N_g = S_g^* \times [0,1]$ for $g = 1, \ldots, 4$ and let us denote by A_g the boundary of N_g which is homeomorphic to an annulus. Let N_0 be a solid torus to which we consider 4 parallel and disjoint annuli $B_1, \ldots, B_4$ whose core curves represent generators of $\pi_1(N_0)$. We let N be the manifold obtained by identifying A_g with B_g for $g = 1, \ldots, 4$, and N' by identifying A_1 with B_1, A_4 with B_4, A_2 with B_3 and A_3 with B_2. Both fundamental groups are isomorphic by van Kampen's theorem. These manifolds are hyperbolizable by Thurston's theorem and one-ended by construction.

We now let γ_j be a non trivial curve in $S_j^* \times \{0\}$ for $j = 1, 2$. Their concatenation defines in N a loxodromic element represented by a boundary curve, so its fixed points are not in the residual set. In N', since the cyclic order is different, this curve is now represented by a loxodromic element whose fixed points cannot lie on the boundary of a component, hence are residual.

7.2. A non-Kleinian group quasi-isometric to a Kleinian group. We recall a construction of Kapovich and Kleiner which provides us with a torsion-free word hyperbolic group which is quasi-isometric to a convex-cocompact Kleinian group but is not isomorphic to a convex-cocompact Kleinian group: this implies that it is necessary, in some cases, to pass to a subgroup of finite index, and that it is not an artifact of the proof.

Following [**KK**], let us consider a 2-torus with two holes $S = \mathbb{T}^2 \setminus (D_1 \cup D_2)$, and let α represent ∂D_1 and $\beta = \partial D_2$. Define X as the complex obtained from S by gluing β to α by a degree 2 covering map $\beta \to \alpha$.

The fundamental group of S is isomorphic to $\mathbb{F}_3 =< a, b, c >$ where we can choose c to be represented by α. The other boundary component can be expressed as $\beta = caba^{-1}b^{-1} = c[a,b]$.

The fundamental group of X is obtained by an HNN-extension of $\pi_1(S)$:

$$\pi_1(X) = \mathbb{F}_3 *_{\mathbb{Z}} =< a, b, c, t | tc^2t^{-1} = c[a, b] > .$$

As such, $G = \pi_1(X)$ is a torsion-free group. It is also word-hyperbolic according to Bestvina-Feighn's combination theorem.

The JSJ decomposition is given by a graph with two vertices and two edges connecting the two vertices. The first vertex corresponds to the fundamental group of S, labeled by $\mathbb{F}_3$, the second to a cylinder with cyclic fundamental group $\mathbb{Z} =< \gamma >$. Note that $\mathbb{F}_3$ corresponds to a surface group in this decomposition.

The first edge identifies one boundary component to α, yielding the morphism $c \mapsto \gamma$, and the second edge identifies the other boundary component to β^2, yielding $c[a, b] \mapsto \gamma^2$.

The limit set of c splits ∂G into three components, and its action exchanges the two components attached at β and keep fixed the third one. Therefore, this decomposition is not regular. It turns out that G cannot be a 3-manifold group because of the identification $\alpha^2 = \beta$ which should take its origin in an essential annulus or Möbius band.

Nonetheless G contains a finite-index subgroup G' with a regular JSJ-decomposition. This can be seen topologically by considering a finite covering $S' \to S$ which has degree 1 over α, degree 2 over β and such that there are twice the number of preimages of α than of β.

For the group G', we may build an irreducible, orientable, Haken 3-manifold with fundamental group G'. See [**KK**, § 8] for more details.

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