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Actions of quasi-Möbius groups

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Abstract

This article is a survey on actions of quasi-Möbius groups. We focus on their applications to Riemannian geometry —in particular hyperbolic geometry— and to the geometry of metric compact spaces. We prepare these applications by providing ample background on quasi-Möbus maps and quasiconformal geometry.

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1 Introduction

A *quasi-Möbius map* is an injective continuous map $h : X \to Y$ between metric spaces X and Y supplied with a homeomorphism $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ —a distortion function— such that, for any distinct points $x_1, x_2, x_3, x_4 \in X$,

$$[h(x_1) : h(x_2) : h(x_3) : h(x_4)] \leq \theta([x_1 : x_2 : x_3 : x_4])$$

where

$$[x_1 : x_2 : x_3 : x_4] = \frac{|x_1 - x_2| \cdot |x_3 - x_4|}{|x_1 - x_3| \cdot |x_2 - x_4|}$$

denotes the *metric cross-ratio*. The map h is more precisely a θ-quasi-Möbius map. When $\theta = \mathrm{Id}$, then h is a *Möbius* map.

This class was explicitly defined and studied by J. Väisälä [Väi2], but they have appeared implicitly before, for instance in G. Mostow's work on rigidity and in J. Ferrand's work on the group of conformal transformations of Riemannian manifolds.

The main objectives of this survey are to show that this is a natural class of groups, that they appear in many different situations and that they have stimulated interesting research. There are several reasons which make these groups appealing and useful. Here is a sample.

- Quasi-Möbius maps have different facets: they are very close to different classes of maps, in particular quasiconformal and quasisymmetric maps so that they have interesting infinitesimal properties and global ones as well. They have the advantage of being defined by global properties compared to the former, and to have richer dynamics compared to the latter.

- They form an intermediate class between the more rigid class of conformal maps and the too flexible class of homeomorphisms.

- Since they are defined in metric terms, quasi-Möbius maps can be considered on very general classes of spaces.

These groups play at least three main roles.

- They capture the global behavior of conformal mappings on compact manifolds.

- They are traces at infinity of isometries (and more generally quasi-isometries) of hyperbolic spaces, so they capture the large scale geometry of groups acting by isometries on hyperbolic spaces.

- A compact metric space which supports a large group of quasi-Möbius mappings enjoy self-similarity features.

This survey will address the following questions and problems, and develop the following aspects.

1. Determine when an action of a group of homeomorphisms is conjugate to that of a group of quasi-Möbius maps, and when the latter is conjugate to a group action of Möbius maps.
2. Given a metric space, describe the group of quasi-Möbius maps.
3. Classify metric spaces with large groups of quasi-Möbius maps.
4. Use quasi-Möbius group actions to describe the geometry of the spaces on which they act.

Outline of the paper.— In the next section, we recall the main properties of quasi-Möbius mappings between Riemannian manifolds and provide several characterizations from different points of view. We also introduce some fundamental tools: the notion of moduli of family of curves which are the most powerful conformal invariants.

In Section 3, we study the dynamics of quasi-Möbius mappings —these are convergence actions, and we review some results characterizing quasi-Möbius groups in dynamical terms. We also recall the Hausdorff-Gromov convergence of metric spaces which enables to apply compactness arguments.

Section 4 focuses on actions on Riemannian manifolds. After adding some specific properties, we address the question of determining when a quasi-Möbius group is conjugate to a Möbius group. We show the role of quasi-Möbius maps in J. Ferrand's solution to the Lichnerowicz conjecture. We also consider groups on non-compact manifolds. In some cases, we show how conformal invariants can be used to obtain a metric for which quasi-Möbius maps become bi-Lipschitz. The last paragraph is concerned with the quasiconformal homogeneity of manifolds. It measures in some sense the size of the group of quasiconformal mappings.

Section 5 is devoted to actions on metric spaces. We describe quantitative geometric properties of metric spaces which are preserved under quasi-Möbius maps. In particular, we introduce the conformal gauge of a metric space and its conformal dimension. We also provide metric characterizations of quasi-Möbius deformations of the triadic Cantor set, the circle and more generally of spheres. Loewner spaces are introduced. They form an important class of metric spaces on which quasi-Möbius maps retain most of their properties. It will enable us to develop differential calculus in the sense of Cheeger, and to generalize some results from the Riemannian setting. We also study properties of compact metric spaces which carry a large group of uniform quasi-Möbius maps. We discuss these issues in the context of subRiemannian manifolds and draw some rigidity phenomena.

The relationships between quasi-Möbius groups and hyperbolicity are discussed in Section 6. We make the correspondence between quasi-Möbius maps and quasi-isometries. This implies that quasi-Möbius invariants are quasi-isometry invariants. Several applications are given, in particular to the classification of negatively curved Riemannian manifolds. We focus on rank one symmetric spaces, homogeneous manifolds of negative curvature and low dimensional manifolds.

Section 7 deals with actions on functional spaces. Under suitable conditions on metric spaces, we can adapt a large class of real-valued functional spaces appearing in harmonic analysis and study their invariance under quasi-Möbius maps. It turns out that these spaces characterize conformal gauges —under suitable conditions. Their relationship to ℓ^p-cohomology is also discussed: this is an interesting instance where hyperbolic methods help understand the quasiconformal geometry of compact spaces.

The last section of the paper is devoted to actions of quasi-Möbius groups on compact sets homeomorphic to the Sierpiński carpet. This compact set has a very large group of homeomorphisms and we show that, depending on the gauge, the group of quasi-Möbius maps may have very different nature.

Conventions and notation — If X denotes a metric space, we will write the distance between two points $x, y \in X$ by one of the following expressions $d(x,y) = d_X(x,y) = |x-y|$. The open ball centred at x and of radius $r > 0$ is denoted by $B = B(x,r) = \{y \in X,\ |x-y| < r\}$ and the closed ball $\overline{B(x,r)} = \{y \in X,\ |x-y| \leq r\}$. Note that the closed ball may be larger than the closure of the open ball. Given $\lambda > 0$, λB denotes the concentric ball of radius λr: $\lambda B = B(x, \lambda r)$. The diameter of a set $E \subset X$ is $\operatorname{diam} E = \sup\{|x-y|,\ x, y \in E\}$.

A metric space X is *proper* if its compact subsets correspond to the closed and bounded subsets of X. A measure μ will always be a Borel regular measure which gives positive mass to any non-empty open subset and finite mass to any bounded set. In particular, if X is proper then the measure is a Radon measure.

General bibliography. — There exist many surveys and books on different parts of this text, which are usually more detailed. We list here a few. Background on analysis on metric spaces includes [Hen, HKST2]. Hyperbolic metric spaces and groups are developed for instance in [Gro3, CDP, GdlH]. The relationships between quasiconformal and hyperbolic geometries are discussed in particular in [KB, BP3, Bon, Klr, Haï2, McT] and in the survey [Bou7] included in this volume. Functional spaces invariant under quasi-Möbius mappings will be discussed in the forthcoming survey [KSS].

Warning. — The topics discussed here correspond to the author's tastes and are limited by the author's knowledge. This holds —unfortunately— for the references as well. Several parts of the paper concerning rigidity phenomena in hyperbolic geometry related to quasiconformal geometry are common with M. Bourdon's survey. It is my pleasure to refer to his article, especially since it is more detailed in many places [Bou7].

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2 Properties of quasi-Möbius maps

Let X be a metric space and x_1, x_2, x_3 and x_4, with $x_1 \neq x_3$ and $x_2 \neq x_4$. Denote by

$$\kappa = [x_1 : x_2 : x_3 : x_4] = \frac{|x_1 - x_2| \cdot |x_3 - x_4|}{|x_1 - x_3| \cdot |x_2 - x_4|}$$

the cross-ratio of the four points. Note that if $x_1 = x_4$ or $x_2 = x_3$, then $\kappa = 1$. Recall that, given a distortion function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$, a θ-quasi-Möbius map is an injective continuous map $h : X \to Y$ such that, for any distinct points $x_1, x_2, x_3, x_4 \in X$,

$$[h(x_1) : h(x_2) : h(x_3) : h(x_4)] \leq \theta([x_1 : x_2 : x_3 : x_4])$$

When θ is a linear function, then h is easily seen to be locally bi-Lipschitz.

We first provide an interpretation of cross-ratios which is easier to handle [BnK1, Lemma 2.3].

Lemma 2.1 (Bonk & Kleiner)**.** *Let X be a metric space. If x_1, x_2, x_3, x_4 are four distinct points of X, let us consider*

$$\langle x_1, x_2, x_3, x_4 \rangle = \frac{\min\{|x_1 - x_2|, |x_3 - x_4|\}}{\min\{|x_1 - x_3|, |x_2 - x_4|\}}\,.$$

Then

$$\langle x_1, x_2, x_3, x_4 \rangle \leq \eta_0([x_1, x_2, x_3, x_4]) \quad \text{and} \quad [x_1, x_2, x_3, x_4] \leq \eta_1(\langle x_1, x_2, x_3, x_4 \rangle)$$

where

$$\eta_0(t) = t + \sqrt{t^2 + t} \quad \text{and} \quad \eta_1(t) = t(2 + t)\,.$$

Most of the properties of quasi-Möbius maps may be established using this version of the cross-ratio.

2.1 Completion and unbounded spaces

Möbius transformations of the Riemann sphere are the typical examples of quasi-Möbius maps: in particular, we may observe that this class of maps do not preserve completeness, cf. $z \in \mathbb{C} \setminus \{0\} \mapsto 1/z \in \mathbb{C} \setminus \{0\}$.

Let us first introduce the one-point compactification of a metric space which provides us with a first collection of examples of quasi-Möbius maps between metric spaces.

Proposition 2.2. *We have the following two inverse constructions. Let $\theta(t) = 16t$.*

- *Let (Z, w) be an unbounded pointed metric space. There exists a metric $\widehat{d}$ on the one-point compactification $\widehat{Z} = Z \cup \{\infty\}$ such that* $\mathrm{Id} : (Z, d) \to (\widehat{Z}, \widehat{d})$ *is θ-quasi-Möbius and*

$$\frac{1}{4} \frac{d(x, y)}{(1 + d(x, w))(1 + d(y, w))} \leq \widehat{d}(x, y) \leq \frac{d(x, y)}{(1 + d(x, w))(1 + d(y, w))}$$

with the convention that $d(x,\infty)=\infty$ *and* $d(x,\infty)/(1+d(y,\infty))=1$*. If* Z *is complete then* $\widehat{Z}$ *is complete.*

- *Let* (X,d,w) *be a marked metric space. There exists a metric* d_w *on* $X\setminus\{w\}$ *such that* $\mathrm{Id}:(X\setminus\{w\},d)\to(X\setminus\{w\},d_w)$ *is* θ*-quasi-Möbius and*

$$\frac{1}{4}\frac{d(x,y)}{\delta(x)\delta(y)}\le d_w(x,y)\le\frac{d(x,y)}{\delta(x)\delta(y)}$$

where $\delta(x)=d(x,w)$*. If* (X,d) *is bounded and complete, then* $(X\setminus\{w\},d_w)$ *is also complete.*

The first part is [BnK2, Lemma 2.2] and the second [Haï3, Lemma 4.5].

Let $f:X\to Y$ be a quasi-Möbius mapping and let us consider X and Y as subsets of $\widehat{X}$ and $\widehat{Y}$. It can be checked using the definitions of cross-ratios that the image of any Cauchy sequence in $\widehat{X}$ is also a Cauchy sequence in $\widehat{Y}$ and that two Cauchy sequences in $\widehat{X}$ are equivalent if and only if their images are equivalent Cauchy sequences. It follows that, adding the points at infinity if necessary, we may always assume that a quasi-Möbius map is defined in a complete metric space.

We conclude this section with the following corollary that can obtained by composing both constructions.

Corollary 2.3. *There are a constant* D *and a distortion function* $\theta:\mathbb{R}_+\to\mathbb{R}_+$ *with the following property. Let* X *be a (complete) metric space and* $\tau=(x_1,x_2,x_3)$ *be three distinct points of* X*; there exists a (complete) bounded metric* d_τ *such that* $\mathrm{Id}:(X,d)\to(\widehat{X},d_\tau)$ *is* θ*-quasi-Möbius and*

$$1/D\le\frac{d_\tau(x_i,x_j)}{\mathrm{diam}_{d_\tau}\widehat{X}}\le D$$

where $i,j\in\{1,2,3\}$ *and* $i\ne j$*.*

2.2 Quasisymmetric maps

We may observe that Möbius maps —hence quasi-Möbius maps as well— do not enjoy global and uniform bounded distortion properties. We introduce the following class of maps to describe their geometric properties. A homeomorphism $h:X\to Y$ between metric spaces is called *quasisymmetric* provided there exists a homeomorphism $\eta:[0,\infty)\to[0,\infty)$ such that $d_X(x,a)\le td_X(x,b)$ implies $d_Y(f(x),f(a))\le\eta(t)d_Y(f(x),f(b))$ for all triples of points $x,a,b\in X$ and all $t\ge0$ [TV1]; see also [Hen].

The basic distortion bound for quasisymmetric maps is given by the following lemma [Hen, Prop. 10.8].

Lemma 2.4. *Let* $h:X\to Y$ *be an* η*-quasisymmetric map between metric spaces. For all* $A,B\subset X$ *with* $A\subset B$ *and* $\mathrm{diam}\,B<\infty$*, we have* $\mathrm{diam}\,h(B)<\infty$ *and*

$$\frac{1}{2\eta\left(\frac{\mathrm{diam}\,B}{\mathrm{diam}\,A}\right)}\le\frac{\mathrm{diam}\,h(A)}{\mathrm{diam}\,h(B)}\le\eta\left(2\frac{\mathrm{diam}\,A}{\mathrm{diam}\,B}\right).$$

Another instance is the following precompactness result.

Theorem 2.5. *Let $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ be a fixed distortion function, $M < \infty$ and X, Y be two metric spaces, $x, x' \in X$. Then*

$$\{f : X \to Y \ \eta\text{-quasisymmetric such that } d(f(x), f(x')) \leq M\}$$

is equicontinuous. Furthermore, any limit of such functions is either constant or η-quasisymmetric.

We record the following relationships, see [Väi2] for a proof, or apply Lemma 2.1.

Proposition 2.6. *Let $f : Z \to Z'$ be a homeomorphism between proper metric spaces.*

(i) If f is η-quasisymmetric then f is also θ-quasi-Möbius, where θ only depends on η.

(ii) If f is θ-quasi-Möbius, then f is locally η-quasisymmetric, where η only depends on θ.

(iii) Let us assume that f is θ-quasi-Möbius. If Z and Z' are unbounded, then f is θ-quasisymmetric. If Z and Z' are compact, then assume that there are three points $z_1, z_2, z_3 \in Z$, such that $|z_i - z_j| \geq \operatorname{diam} Z/\lambda$ and $|f(z_i) - f(z_j)| \geq \operatorname{diam} Z'/\lambda$ for some $\lambda > 0$, then f is η-quasisymmetric, where η only depends on θ and λ.

We may deduce the following behavior of sequences of quasi-Möbius maps from Proposition 2.6 together with Theorem 2.5.

Corollary 2.7. *Let us consider a family of uniformly quasi-Möbius embeddings $f_n : X \to Y$ between compact metric sets X, Y. Pick distinct points $x_1, x_2, x_3 \in X$.*

- *either there is a subsequence $(f_{n_k})_k$ and a positive size $\delta > 0$ such that $d(f_{n_k}(x_p), f_{n_k}(x_q)) \geq \delta$ for all $p \neq q \in \{1, 2, 3\}$ and $(f_{n_k})_k$ is equicontinuous,*
- *or there is a subsequence $(f_{n_k})_k$ and points $a \in X$ and $b \in Y$ such that $(f_{n_k})_k$ is uniformly convergent to the constant map $\{b\}$ on compact subsets of $X \setminus \{a\}$.*

2.3 Moduli of curves

We introduce in this paragraph one of the most powerful tools to analyze and understand quasi-Möbius mappings first in Euclidean spaces, and then in more general settings, cf. Section 5.2. This notion of moduli of curves was inspired by electricity and one can find its roots in J.C. Maxwell's work. Its first form was defined by A. Beurling in his thesis, and then developed with L. Ahlfors [AB] under the form of *extremal length*. It was modified to its actual form, inspired by the work of B. Fuglede and C. Loewner [Fug, Loe]. This point of view was the starting point of J. Heinonen and P. Koskela's work in general metric spaces [HnK2]. A general account is given in [Hen].

What makes moduli so useful is the conjunction of the two following phenomena. On the one hand, they are quasi-invariant under quasi-Möbius maps, and on the other hand, they encode geometric properties of the spaces.

Rectifiable curves.— Let (X, d) be a metric space. A curve γ in X is a continuous map $\gamma : I \to X$ where I is an interval in $\mathbb{R}$. Most of the time, we identify γ with its image $\gamma(I)$. If $I = [a, b]$ is a closed interval (in this case we say that γ is a closed curve), the length of $\gamma : I \to \mathbb{R}$ is defined by

$$\ell(\gamma) = \sup \sum_{i=1}^{N-1} d(\gamma(t_i), \gamma(t_{i+1}))$$

where the supremum is taken over all subdivisions $a = t_1 \leq t_2 \leq .... \leq t_N = b$ of I. If I is not a closed interval, the length of γ is defined by $\ell(\gamma) = \sup \ell(\gamma(J))$ where the supremum is taken over all closed subintervals J of I (and so $\gamma(J)$ is a closed subcurve of γ). A curve with finite length is *rectifiable*. It is *locally rectifiable* if all its closed subcurves are rectifiable. Any rectifiable curve $\gamma : I \to X$ admits a decomposition of the form $\gamma = \gamma_s \circ s_\gamma$ where $s_\gamma : I \to [0, \ell(\gamma)]$ is the length function and $\gamma_s : [0, \ell(\gamma)] \to X$ is the unique 1-Lipschitz map so that such a decomposition holds. The curve/map γ_s is called the arclength parameterization of γ. In this case, if $g : X \to [0, +\infty]$ is a Borel function, the integral of g along γ is defined by

$$\int_\gamma g(s) ds = \int_0^{\ell(\gamma)} g \circ \gamma_s(t) dt.$$

If γ is locally rectifiable, then

$$\int_\gamma g(s) ds = \sup \int_{\gamma'} g(s) ds$$

where the supremum is taken over all rectifiable subcurves γ' of γ.

Families of curves.— Let (X, d, μ) be a metric measure space. Let Γ be a curve family in X and let $p \geq 1$. We define the p-modulus of Γ by

$$\mathrm{mod}_p(\Gamma) = \inf \int_X \rho^p d\mu$$

where the infimum is taken over all Borel functions $\rho : X \to [0, +\infty]$ such that $\int_\gamma \rho(s) ds \geq 1$ for any locally rectifiable curve $\gamma \in \Gamma$. Such a ρ is called *admissible* for the curve family Γ.

Remark 2.8. *1. The p-modulus of the family Γ of all curves that are not locally rectifiable is zero, since any function $\rho : X \to [0, +\infty]$ is admissible for Γ.*

2. *If Γ contains a constant curve, then there is no admissible function for Γ, and (by convention) $mod_p(\Gamma) = +\infty$.*

Theorem 2.9 (Properties of the modulus)**.** *Let (X, d, μ) be a metric space and let $p \geq 1$.*

(i) $\mathrm{mod}_p(\emptyset) = 0$*;*

(ii) If Γ_1, Γ_2 are two curve families in X with $\Gamma_1 \subset \Gamma_2$, then $\mathrm{mod}_p(\Gamma_1) \leq \mathrm{mod}_p(\Gamma_2)$*;*

(iii) If $(\Gamma_i)_{i\in\mathbb{N}}$ is a countable collection of curve families in X, then $\mathrm{mod}_p(\cup_i \Gamma_i) \leq \sum_i \mathrm{mod}_p(\Gamma_i)$*;*

(iv) If Γ, $\tilde{\Gamma}$ are two curve families in X such that each curve $\gamma \in \Gamma$ has a subcurve $\tilde{\gamma} \in \tilde{\Gamma}$, then $\mathrm{mod}_p(\Gamma) \leq \mathrm{mod}_p(\tilde{\Gamma})$*;*

(v) Let $(\Gamma_n)_n$ be an increasing sequence of families of curves. Then, for $p > 1$, $\mathrm{mod}_p(\cup\Gamma_n) = \lim \mathrm{mod}_p\Gamma_n$*.*

Note that properties (i), (ii) and (iii) imply that the p-modulus is an outer measure on the set of all curves in X (but in general there is no nontrivial measurable family of curves !).

Condensers and capacities.— A *condenser* is given by two disjoint compact connected sets (E, F). Given a condenser (E, F), let $\Gamma(E, F)$ denote the set of curves of joining E and F. The *p-capacity* of (E, F) is defined by

$$\mathrm{cap}_p(E, F) = \mathrm{mod}_p\Gamma(E, F)\,.$$

2.4 Analytic properties

Given $K \geq 1$, a *K quasiconformal map* $f : (M, g) \to (N, h)$ between Riemannian manifolds of dimension $n \geq 2$, is a homeomorphism which belongs to the local Sobolev space $W^{1,n}_{loc}(M, N)$ and such that

$$\|Df\|^n \leq KJ_f \quad \text{a.e.}$$

where Df denotes the differential of f in the sense of distributions, and J_f denotes its Jacobian determinant.

Ferrand cross-ratios and quasi-Möbius maps.— Given four distinct points $x_1, \ldots, x_4$ in a Riemannian manifold (M, g) of dimension $n \geq 2$, define

$$[x_1 : x_2 : x_3 : x_4]_F = \inf_{(E,F)} \mathrm{cap}_n(E, F)$$

where the infimum is taken over all condensers (E, F) such that $\{x_1, x_2\} \subset E$ and $\{x_3, x_4\} \subset F$. This cross-ratio is continuous on the set of distinct quadruples, takes an infinite value if and only if $\{x_1, x_2\} \cap \{x_3, x_4\} \neq \emptyset$ and vanishes if and only if $x_1 = x_2$ or $x_3 = x_4$ [Fer3, Fer6].

Say that an embedding $f : (M, g) \to (N, h)$ is an *F-quasi-Möbius map* if there exists a distortion function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$[f(x_1) : f(x_2) : f(x_3) : f(x_4)]_F \leq \theta([x_1 : x_2 : x_3 : x_4]_F)$$

for all quadruples.

Theorem 2.10. *Let $f : M \to N$ be a homeomorphism between Riemannian manifolds of dimension $n \geq 2$. The following are equivalent.*

1. *f is an F-quasi-Möbius map;*
2. *f is quasiconformal;*
3. *f preserves n-moduli of curves up to a fixed factor.*

If f is quasi-Möbius, then f satisfies all the above properties. When the manifolds are compact, this is an equivalence. When the manifolds are not, then it is an equivalence if and only if there are increasing homeomorphisms $\eta_\pm : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\eta_-([x_1, \ldots, x_4]) \leq [x_1, \ldots, x_4]_F \leq \eta_+[x_1, \ldots, x_4])$$

for all quadruples.

The equivalence between the quasiconformal and the quasi-Möbius properties holds in particular for Riemannian manifolds (M, g) of dimension n with non-negative Ricci curvature and upper volume bound $vol_g(B(x, r)) \lesssim r^n$ [HnK2, § 6.5].

Moreover the following properties hold.

Theorem 2.11. *A quasi-Möbius map between Riemannian manifolds of dimension $n \geq 2$ is*

1. *absolutely continuous i.e., it preserves sets of measure zero;*
2. *absolutely continuous on almost every curves in the sense that the n-modulus of the family of curves on which the restriction of f is not absolutely continuous is zero;*
3. *differentiable almost everywhere.*

These properties were first established in dimension 2 after a long succession of steps. L. Ahlfors, L. Bers, F. Gehring, A. Mori, A. Pfluger, K. Strebel have contributed to this problem. In dimension 3, this is attributed to F. Gehring and J. Väisälä. They have then been generalized to arbitrary dimension by G.D. Mostow. The differentiability of quasiconformal homeomorphisms follows from Stepanov's theorem in arbitrary dimension (an argument due to F. Gehring and O. Lehto enables to bypass this result in dimension 2). Standard background on these topics include [Ahl, Väi1].

A *conformal diffeomorphism* is a diffeomorphism between Riemannian manifolds $f : (M, g) \to (N, h)$ such that $f^*h = e^u g$ for some smooth function u. The standard Liouville theorem asserts that a conformal diffeomorphism between open sets of Euclidean spaces of dimension at least three is the restriction of a Möbius transformation of the extended space. In the realm of quasiconformal geometry, the Liouville theorem is concerned with the characterization of conformal diffeomorphisms among quasiconformal maps.

Theorem 2.12 (Ferrand [Fer4])**.** *A homeomorphism between Riemannian manifolds is a conformal diffeomorphism if and only if it is* 1*-quasiconformal.*

This theorem in the context of open subsets of Euclidean spaces follows from Weyl's lemma in dimension 2, and is due to F. Gehring [Geh] and Y. Reshetnyak [Res] in arbitrary dimension. Let us mention the short proof of P. Tukia and J. Väisälä that a 1-quasiconformal homeomorphism of $\mathbb{S}^n$, $n \geq 2$, is conformal using their compactness properties and the fact that they form a group [TV2].

Remark 2.13. *In dimension* 1*, quasi-Möbius maps and quasisymmetric maps need not be absolutely continuous. Quasisymmetric maps are characterized by the fact that their derivative in the sense of distributions is a doubling measure* μ *i.e., there exists a constant* $C \geq 1$ *such that* $\mu((x-2t, x+2t)) \leq C\mu((x-t, x+t))$ *for all* $x \in \mathbb{R}$ *and* $t > 0$*, cf. [Tuk1].*

2.5 Topological characterizations of quasi-Möbius maps

We provide a characterization in the spirit of [BA, Th. 2], [Geh, Th. 18 and Cor. 8], [Pan2, Prop. 43] and [Pau, § 3]; see also [AH, Cor. 1].

Say a family of pointed functions $\mathcal{F} = \{(X, x) \xrightarrow{f} (Y, y)\}$ is *bi-equicontinuous* if

1. for all $\varepsilon > 0$, there is some $\alpha > 0$ such that, for any $f \in \mathcal{F}$, if $d(x, z) \leq \alpha$ then $d(f(x), f(z)) \leq \varepsilon$ and
2. for all $\alpha > 0$, there is some $\varepsilon > 0$ such that, for any $f \in \mathcal{F}$, if $d(x, z) \geq \alpha$ then $d(f(x), f(z)) \geq \varepsilon$.

We associate to $\mathcal{F}$ the collection of *normalized* maps $\mathcal{N}_{\mathcal{F}}$ as follows: for each $(X \xrightarrow{f} Y) \in \mathcal{F}$ and each triple $\tau = (a, b, c) \in X^3$, we associate the normalized map (f_τ, a) by applying Corollary 2.3 to $(X, (a, b, c))$ and to $(Y, (f(a), f(b), f(c)))$ and by letting $f_\tau = ((X, d_\tau) \xrightarrow{f} (\widehat{Y}, d_{f(\tau)})$.

Proposition 2.14. *Let* $\mathcal{F}$ *be a collection of embeddings. There exists* θ *such that* $\mathcal{F}$ *is a family of* θ*-quasi-Möbius mappings if and only if* $\mathcal{N}_{\mathcal{F}}$ *is bi-equicontinuous.*

Remark 2.15. *This criterion also applies when* $\mathcal{F}$ *contains a single map.*

3 The topological dynamics of uniformly quasi-Möbius groups: convergence actions

A *convergence group action* is an action of a group G on a metrizable compact space X with the following property: any sequence of distinct elements $(g_n)_n$ of G

contains a subsequence $(g_{n_k})_k$ which satisfies one of the following two properties:

1. either the sequence $(g_{n_k})_k$ is uniformly convergent to a homeomorphism g of X;

2. or there exist $a, b \in X$ such that $(g_{n_k})_k$ is a *collapsing sequence of base* (a, b) i.e., this subsequence is convergent to the constant map $\{b\}$ uniformly on compact subsets of $X \setminus \{a\}$,

A convergence group is *discrete* if case 1. above never occurs. We shall say that a discrete convergence group is *uniform* if its action is cocompact on the set of distinct triples.

Remark 3.1. *If X is complete and is non-compact, then any homeomorphism fixes the point at infinity in $\widehat{X}$. Thus, any collapsing sequence of a convergence group on $\widehat{X}$ has to contain the point at infinity in its base. If X has at least three ends, then this is impossible, cf. [Fer7, Appendix]; in such a setting, convergence groups are all compact.*

Corollary 2.7 implies at once:

Theorem 3.2. *A group of uniformly quasi-Möbius self-maps of a compact metric space has the convergence property. In Riemannian manifolds, uniformly F-quasi-Möbius groups have also the convergence property.*

An interesting question is whether the converse holds as well:

Question 3.3. *If G is a convergence group acting on a metrizable space, is there a metric on X compatible with its topology which turns G into a uniform quasi-Möbius group?*

We will provide partial answers in the last paragraph of this section. We briefly review some properties of convergence actions. In order to analyze the dynamics of quasi-Möbius groups, we will recall the definition of Hausdorff-Gromov convergence.

The *limit set* Λ_G *of* G is the set of points a which belongs to the base of a collapsing sequence; it is a compact G-invariant subset of X. A convergence group G is said to be *non-elementary* if Λ_G has at least three points. In this case, Λ_G is a perfect set and the action of G on Λ_G is minimal (every orbit is dense).

When G is discrete, the complement $\Omega_G = X \setminus \Lambda_G$ is called the *ordinary set* and corresponds to the set of points $x \in X$ which have a neighborhood V so that $g(V) \cap V \neq \emptyset$ for at most finitely many group elements $g \in G$.

3.1 Topological properties

We record some properties which follow from the existence of a uniform convergence action.

We mention a dynamical property which shows how rich is the dynamics of uniform convergence groups. This is reminiscent of D. Sullivan's notion of expanding covers [Sul3].

Proposition 3.4 (topological transitivity)**.** *Let G be a uniform convergence group acting on a compact space X. For any open subset $U \subset X$, there exist finitely many elements $g_1, \ldots, g_n \in G$ such that*

$$X = \bigcup_{1\le j\le n} g_j(U)\,.$$

This proposition also holds for discrete convergence groups acting on their limit set.

Corollary 3.5. *Let G be a uniform convergence group action acting on a compact space X. All open sets have the same topological dimension.*

The following is a very strong theorem.

Theorem 3.6 (Swarup [Swa])**.** *If X is a connected metrizable space which admits a uniform convergence group action, then X is locally connected and contains no cut points.*

Low dimensional compact spaces are classified as follows:

Theorem 3.7 (Kapovich & Kleiner [KK])**.** *Let X be a one-dimensional compact, connected, metrizable space which admits a uniform convergence action. One of the following exclusive cases occurs.*

- *X is homeomorphic to the Sierpiński carpet, or*
- *X is homeomorphic to the Menger sponge, or*
- *X admits local cut points.*

All these cases exist. Let us remark that both the Sierpiński carpet and the Menger sponge are one-dimensional, connected, locally connected, metrizable, compact spaces, with no local point. Their only difference is that the former is planar whereas the latter has no planar open subsets. All these properties characterize both sets up to homeomorphisms. Let us also mention that they are both universal in the following sense: any one-dimensional connected metrizable compact space can be embedded in the Menger sponge; if it is planar, then it can also be embedded in the Sierpiński carpet.

The idea of the proof of the theorem is to start with a compact set X with no local cut point and assume that it contains a planar open subset. The dynamics enables them to prove that the whole set is planar; more precisely, they show that any embedded graph has to be planar by pushing it into the planar open set, thanks to the topological transitivity. Then Claytor's planarity theorem implies that the compact set is planar, hence homeomorphic to the Sierpiński carpet [Cly]. The other case is that X has no planar open subset, hence is homeomorphic to the Menger sponge.

The following result takes its roots in [Fer7].

Proposition 3.8. *Let G be a non-elementary convergence group acting on a compact metrizable space X. If there is some point $x \in \Lambda_G$ which has a neighborhood homeomorphic to $\mathbb{R}^n$ for some $n \geq 1$, then X is homeomorphic to $\mathbb{S}^n$.*

The basic examples of convergence groups are Möbius groups of the closed unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$. Its discrete subgroups are *Kleinian groups.*

Notes. — Convergence groups were introduced by F. Gehring and G. Martin [GehM] for groups acting on a sphere. Their definition was then generalized to broader settings and systematically studied in particular by P. Tukia [Tuk4] and B. Bowditch [Bwd2].

3.2 Hausdorff-Gromov limits of metric spaces

Let Z be a compact metric space of diameter D. For any $k \geq 1$, let N_k be the minimum number of balls of radius $D/2^k$ needed to cover X.

Let us define inductively $A_0 = X$ and $A_{k+1} = A_k \times [1, N_{k+1}]$ and let $p_k : A_{k+1} \to A_k$ denote the canonical projection.

Let us construct inductively a sequence $(\mathcal{B}_k)_{k\geq 0}$ of covers of Z by balls of radius $D/2^k$. The initial cover $\mathcal{B}_0$ is given by a ball of radius D. If $(B_\alpha)_{\alpha\in A_k}$ is the cover $\mathcal{B}_k$ defined at the next generation, then we cover each ball B_α de $\mathcal{B}_k$ with $N_{k=1}$ balls $B(x_\beta, D/2^{k+1})$, $\beta \in A_{k+1}$, with $p_{k+1}(\beta) = \alpha$. We then let $\mathcal{B}_{k+1} = \{B(x_\beta, D/2^{k+1})\}_{\beta\in A_{k+1}}$.

We may encode the points of Z as follows. Set

$$A = \varprojlim (A_k, p_k) = \left\{ (\alpha_k) \in \prod_{k\geq 0} A_k,\ p_k(\alpha_{k+1}) = \alpha_k \right\} .$$

This is a compact space on which we may define an ultra-metric d_A by letting

$$d_A((\alpha_k), (\beta_k)) = 2^{-\max\{j,\ \alpha_j=\beta_j\}} .$$

Let us define $\pi_Z : A \to Z$ where we identify a sequence in A by a geodesic ray $(B_{\alpha_k})_{\alpha_k\in A_k}$ such that $(\alpha_k) \in A$. One may check that $\pi_Z : A \to Z$ is surjective and $2D$-Lipschitz.

If X and Y are two metric spaces, we write

$$d_{HG}(X, Y) = \inf d_H(f(X), g(Y))$$

where the infimum is taken over all isometric embeddings $f : X \to Z$ and $g : Y \to Z$ in a common metric space Z. This defines a metric on the set of non-empty compact metric spaces up to isometry. The set of non-empty compact spaces which can be encoded by the same space A is compact: each metric space Z can be isometrically embedded in the space of 1-Lipschitz functions $f : Z \to \mathbb{R}$ *via* the map $z_0 \mapsto (z \mapsto (|z - z_0|))$. We thus have an isometric embedding ι_Z from Z into the compact space $\mathrm{Lip}_{2D}(A)$ (Arzéla-Ascoli theorem). The convergence of spaces boils down to the convergence of functions.

Similarly, if $(X_n)_n$ is a sequence of compact metric spaces encoded by a space A, $(Y_n)_n$ by B, and if $f_n : X_n \to Y_n$ is a sequence of functions, the convergence of (f_n) boils down to the convergence of $\iota_{Y_n} \circ f_n \circ \pi_{X_n} : A \to \mathrm{Lip}_{2D}(B)$, where D is an upper bound of the diameters of (Y_n).

A sequence of pointed metric spaces $(X_n, x_n)_n$ tends to (X, x) if the sequence of pointed closed balls $(\overline{B_{X_n}(x_n, R)}, x_n)$ tends to a pointed metric space (X_R, x_R) isometric to a subset of $(\overline{B_X(x, R)}, x)$ containing the open ball $B_X(x, R)$ for all $R > 0$.

A metric space X is N-*doubling* if any set E of diameter D can be covered by at most N sets of diameter $D/2$. In this case, we may define A by considering $N_k = N$ for all $k \geq 0$. M. Gromov proves [Gro2]

Theorem 3.9. *Let (X_n, x_n) be a sequence of proper N-doubling metric spaces. There exists a subsequence (n_k) and a metric space (X, x) such that (X_{n_k}, x_{n_k}) tends vers (X, x).*

This allows us to define the notion of *tangent spaces* for a proper doubling metric space Z: it consists of any limit of $(Z, z_n, R_n d_Z)$ where (z_n) is a sequence of points of Z and $(R_n)_n$ is a positive sequence tending to $+\infty$.

3.3 The conformal elevator principle

For the purpose of this survey, we introduce the following terminology.

Definition 3.10 (large group of quasi-Möbius mappings)**.** *A compact metric space X has a large group of quasi-Möbius maps if there are a constant $m > 0$ and a distortion function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any distinct $x, y, z \in X$, there exists a θ-quasi-Möbius map $f : X \to X$ such that $\{f(x), f(y), f(z)\}$ is an m-separated subset of X.*

Let us remark that a group of uniform quasi-Möbius mappings which acts cocompactly on the set of distinct triples is a large group of quasi-Möbius mappings.

Let X be a doubling compact metric space and assume that there exists $x \in X$, a sequence of θ-quasi-Möbius maps $g_n : X \to X$ triples of points $\{x_n, y_n, z_n\}$ tending to x with the property that $d(x_n, y_n) \asymp d(x_n, z_n) \asymp d(y_n, z_n)$ and $(g_n x_n, g_n y_n, g_n z_n)$ are m-separated. Then X admits a weak tangent space T and there exist a point $a \in X$ and a quasi-Möbius map $h : T \to X \setminus \{a\}$. More generally, the topological transitivity can be quantified as follows:

Proposition 3.11 (Conformal elevator principle)**.** *Let X be a compact metric space with a large group of quasi-Möbius maps. There exist definite sizes $r_0 \geq \delta_0 > 0$ and a distortion function η such that, for any $x \in X$, and any $r \in (0, \operatorname{diam} X/2]$, there exists a θ-quasi-Möbius map $g : X \to X$ such that $g(B(x, r)) \supset B(g(x), r_0)$, $\operatorname{diam} B(g(x), r_0) \geq 2\delta_0$ and $g|_{B(x,r)}$ is η-quasisymmetric.*

This principle is very useful to translate qualitative topological properties of X into quantitative ones as we will see later on. It is also responsible for the quasiselfsimilarity of such compact sets.

Notes.— The relationship of a tangent space and a punctured compact space appears in [BnK2]. The conformal elevator principle takes its roots in D. Sullivan's work on Kleinian groups and rational maps [Sul3]; this formulation is borrowed from [Haï2, Prop. 4.6].

3.4 Convergence actions versus quasi-Möbius maps

There are very few results that ensure that a convergence group action can be realized as the action of uniform quasi-Möbius maps.

We provide one general theorem for groups acting on a circle:

Theorem 3.12 (Casson & Jungreis, Gabai [CJ, Gab])**.** *Let G admit a discrete convergence action on a circle. Then the action of G is conjugate to a Möbius action on $\mathbb{S}^1$.*

This result has had important consequences on 3-manifolds; following the work of G. Mess [Mes], this implies the so-called torus theorem: *Let M be a closed, irreducible, orientable 3-manifold. If its fundamental group is infinite and is not atoroidal, then either it contains an embedded and essential torus, or it admits a Seifert fibration.* This has enabled one to reduce the whole geometrization conjecture of Thurston to the characterization of hyperbolic and spherical manifolds.

The next result applies to particular convergence actions, but on any metrizable compact metric space.

Theorem 3.13. *Let G be a countable group admitting a discrete convergence action on a metrizable compact space Z. If its diagonal action on the set of distinct pairs is cocompact, then Z admits a metric compatible with its topology so that G acts by uniform quasi-Möbius mappings.*

A weaker version was established by B. Bowditch [Bwd1]. In a somewhat different form, this was proved by A. Yaman [Yam], and in this form by V. Gerasimov [Ger1].

In dimension three and above, M. Freedman and R. Skora construct examples of convergence actions of free groups on spheres S^n, $n \geq 3$, which are not conjugate to any quasi-Möbius action [FS1, FS2]. For these convergence groups, the action is cocompact on the ordinary set, and the limit set is a wild Cantor set i.e., cannot be mapped by a global homeomorphism to a Cantor set on a line. For Schottky groups, all the Cantor sets which arise are always tame (not wild). These groups are constructed as follows: given $n, k \geq 2$, let us consider a collection of $2k$ pairwise disjoint balls $\{B_j, B'_j,\ 1 \leq j \leq k\}$ in $\mathbb{S}^n$, and Möbius transformations g_j which map the interior of B_j to the exterior of B'_j. The group generated by $\{g_j,\ 1 \leq j \leq k\}$ is isomorphic to the free group of rank k, and the limit set is a Cantor set. Moving the balls so that their centers belong to a common circle, we may deduce that these limits sets are tame. Let us remark that the wildness of the Cantor set is not a topological obstruction to be the limit set of a convergence group of Möbius maps. Indeed, S. Matsumoto constructed an example of a free group acting as a conformal convergence group on the sphere $\mathbb{S}^3$ such that the limit set is a wild Cantor set and with a cocompact action on its ordinary set [Mtm, Thm 8.1], improving on a previous work of M. Bestvina and D. Cooper [BC].

Remark 3.14. *Let us note that M. Freedman has given an equivalent statement of the four-dimensional surgery conjecture in terms of convergence actions [Fre] which has certainly motivated his contributions above: any discrete convergence group of S^3 with a Cantor limit set and a cocompact action on its ordinary set extends to a convergence action on the four-dimensional ball.*

In dimension 2, the situation is supposed to be very different: G. Martin and R. Skora conjecture the following statement [MnS, Conj. 6.1] —to be compared with Conjecture 6.23.

Conjecture 3.15 (Martin & Skora)**.** *Let G be a convergence action on S^2. Then there exists a discrete subgroup K of $\mathbb{P}SL_2(\mathbb{C})$ isomorphic to G and a continuous cellular map $\varphi : \widehat{\mathbb{C}} \to S^2$ such that $G \circ \varphi = \varphi \circ K$.*

Let us illustrate the necessity to include cellular maps with the following simple example. Let F be a Fuchsian group and consider its action on the quotient $\widehat{\mathbb{C}}/\overline{\mathbb{D}}$ of the Riemann sphere by identifying every point of the closed invariant unit disk. When the action is uniform, then G. Martin and P. Tukia show that the map can be chosen to be a homeomorphism if it exists [MnT]. Positive cases to this conjecture are established in [MnT, MnS, Haï3]; see also § 6.5.3.

Compact groups acting on the sphere are well understood for a long time; see [Ker, Kol] for a proof.

Theorem 3.16 (Kerékjártó)**.** *Any compact subgroup of homeomorphisms of S^2 is topologically conjugate to a closed subgroup of $O(3)$.*

4 Actions on Riemannian manifolds

This section is devoted to actions on Riemannian manifolds. We address four questions and problems: (a) when is a group of quasi-Möbius mappings conjugate to Möbius group? (b) description of manifolds which carry a large group of conformal diffeomorphisms (the Lichnerowicz conjecture); (c) what can be said of quasi-Möbius maps on non-compact manifolds? (d) description of the quasiconformal homogeneity of manifolds. There are overlaps with [Bou7].

Let (M, g) be a Riemannian manifold of dimension n. Considering the length distance d_g defined by g and its Riemannian volume vol_g, we obtain a metric measure space.

Note that when M is compact but not homeomorphic to a sphere, Proposition 3.8 implies that there may only be relatively compact groups of uniform quasi-Möbius mappings that is, uniform quasisymmetric mappings.

If Γ is a curve family in M, the *conformal modulus* of Γ is defined by

$$\mathrm{Mod}(\Gamma) = \mathrm{mod}_n \Gamma = \inf \int_M \rho^n dvol_g$$

where the infimum is taken over all functions $\rho : M \to [0, +\infty]$ such that $\int_\gamma \rho(s)ds \geq 1$ for any locally rectifiable curve $\gamma \in \Gamma$. The key point is that the conformal modulus is conformally invariant.

Theorem 4.1. *Let M, N be two Riemannian manifolds of dimension n. If $f : M \to N$ is conformal, then $Mod(f\Gamma) = Mod(\Gamma)$ for all curve family Γ in M, where $f(\Gamma) = \{f(\gamma), \gamma \in \Gamma\}$.*

We end this short introduction by the following version of the Hilbert-Smith conjecture:

Theorem 4.2 (Martin [Man2])**.** *If a locally compact group G of quasi-Möbius maps acts faithfully on a Riemannian manifold, then G is a Lie group.*

An extension to quasi-Möbius groups acting on some particular metric measure spaces appears in [Mj].

4.1 Quasi-Möbius groups versus Möbius groups

This section is devoted to the following question asked by F. Gehring and B. Palka [GehP, p. 197].

Question 4.3. *Let $n \geq 1$ and G be a group of uniform quasi-Möbius maps acting on the Euclidean sphere $\mathbb{S}^n$. Is the group G conjugate (by a quasi-Möbius map) to a group of Möbius transformations?*

The situation is very different depending on whether $n = 1, 2$ or $n \geq 3$. In the one-dimensional setting, this was solved in several steps [Hin1, Hin2, Mak].

Theorem 4.4 (Hinkkanen, Markovic)**.** *Any uniformly quasi-Möbius group of homeomorphisms on the unit circle is quasisymmetrically conjugate to a group of Möbius transformations.*

Dimension 2 was settled by D. Sullivan.

Theorem 4.5 (Sullivan)**.** *A group of uniform quasi-Möbius homeomorphisms of $\mathbb{S}^2$ is quasisymmetrically conjugate to a group of Möbius transformations.*

Its proof is outlined in [Sul1]. The main idea is to find an invariant measurable conformal structure under the group and to apply the measurable Riemann mapping theorem. A. Hinkkanen wrote a detailed proof following P. Tukia, addressing also the same question for semigroups of uniformly quasiregular mappings of the 2-sphere. In that case, the answer is not always positive [Hin3].

In higher dimension, the situation is not that definite. We need to introduce the notion of conical points for convergence groups. Let G be a convergence group acting on a metrizable metric space. A point x is *conical* if there are sequences $(x_n)_n$ and $(g_n)_n$ of points in X and elements of G respectively such that $(x_n)_n$ tends to x and, for any $y \in X \setminus \{x\}$, the set of triples $(g_n(x), g_n(y), g_n(x_n))$ remains in a compact subset of the set of distinct triples. For a group of uniform quasi-Möbius maps acting on a sphere, the set of conical points has either full or zero measure.

Theorem 4.6 (Tukia [Tuk3])**.** *A group of uniform quasi-Möbius homeomorphisms of* $\mathbb{S}^n$*,* $n \geq 3$*, is quasisymmetrically conjugate to a group of Möbius transformations provided the set of conical points has positive measure.*

In particular, if the action is cocompact on the set of distinct triples, then every point is conical and the group is conjugate to a group of Möbius transformations [Gro1].

As for Sullivan's theorem, P. Tukia first defines an invariant measurable conformal structure. Then the conformal elevator principle allows him to apply a zooming argument at a conical point where this conformal structure is almost continuous. This enables him to conjugate the initial group to a group of conformal transformations.

Remark 4.7. *Let us mention that examples of uniform quasi-Möbius groups which are not conjugate to Möbius groups have been constructed in all dimension* $n \geq 3$ *by P. Tukia [Tuk2], with refinements by G. Martin, M. Freedman and R. Skora, and V. Mayer [Man1, FS1, FS2, May].*

4.2 The Lichnerowicz conjecture

A. Lichnerowicz asked the question of which compact Riemannian manifolds have a compact group of conformal diffeomorphisms. The final answer was brought by J. Ferrand proving

Theorem 4.8 (Ferrand [Fer1])**.** *Let* (M, g) *be a compact manifold. If* M *is not conformally equivalent to a round sphere, then there is a conformal change of metric so that the group of conformal diffeomorphisms coincides with the group of isometries.*

There are two main steps in the proof. The first step is to prove that if the group of conformal diffeomorphisms is compact, then this group reduces to the group of isometries up to conformal change of the metric. This can be achieved by averaging the pushforwards of the metric with respect to the Haar measure of the group. The other step consists in showing that if this group is not compact, then the manifold is conformally equivalent to the round sphere. For this, J. Ferrand observes that the group of conformal diffeomorphisms is uniformly quasi-Möbius, before this notion was even coined, so that she may use its dynamical properties to prove that it is equivalent to the sphere, cf. Proposition 3.8. This equivalence is obtained from a 1-quasiconformal map, which is conformal according to Theorem 2.12.

For a historical account on the Lichnerowicz conjecture and its generalizations to non-compact complete manifolds, see [Fer8].

4.3 Conformally parabolic versus hyperbolic spaces

Let (M, g) be a non-compact complete manifold of dimension $n \geq 2$. The *conformal capacity* of a compact subset K is $\mathrm{cap}_M K = \inf \mathrm{cap}_n(K, \{\infty\})$ where the condenser is considered in the Alexandrov compactification $\widehat{M}$.

It can be proved that it is always finite, and either it is zero for all compact sets, or it is positive for any non-degenerate continuum.

Definition 4.9. *Say that a non-compact complete manifold M is conformally hyperbolic if there exists a compact set K of positive capacity. Otherwise, M is said to be conformally parabolic.*

Theorem 4.10 (Ferrand)**.** *If M is conformally hyperbolic, then, for any distortion function θ, θ-quasi-Möbius mappings and θ-F-quasi-Möbius mappings form equicontinuous families. In particular, there is a conformally equivalent Riemannian metric on M such that the group of conformal diffeomorphisms agrees with the group of isometries.*

The starting point of the proof is the definition of a conformally invariant metric for conformally hyperbolic manifolds: set

$$\mu_M(x,y) = \inf_E \mathrm{cap}_M E$$

where the infimum is taken over continua containing $\{x,y\}$. This defines a metric which is invariant by the group of conformal diffeomorphisms. For this metric, quasiconformal maps become bi-Lipschitz. This implies that θ-quasi-Möbius maps have no collapsing sequences in M.

Examples of conformally hyperbolic manifolds include simply connected manifolds with pinched negative curvature. More generally, if M satisfies some isoperimetric inequality stronger than in the Eucldiean space, then M is conformally hyperbolic; see [Pan2] for details.

Examples of conformally parabolic manifolds include Euclidean spaces. Up to a conformal change of metrics, this is the only example for which the group of conformal homeomorphisms is not closed; cf Remark 3.1 for the multi-ended situation. This solves the extended version of the Lichnerowicz conjecture.

Theorem 4.11 (Ferrand [Fer5])**.** *Let M be a complete non-compact Riemannian manifold. If M is not conformally equivalent to the Euclidean space, the group of conformal diffeomorphisms can be reduced to a group of isometries by a conformal change of metrics.*

As for the compact case, the basic idea is to prove the existence of a collapsing sequence on $\widehat{M}$, when M is conformally parabolic.

4.4 Homogeneity

The following topic was initiated by F. Gehring and B. Palka [GehP] in the setting of Euclidean domains. We refer to [BTCT] for more detailed and quantitative statements and references. The appropriate regularity assumptions on maps for this topic is quasiconformality. The basic problem is to determine, given a closed subset $E \subset M$ of a manifold M whether the group of quasiconformal mappings of M preserving E admits a transitive action or not i.e., for any $x, y \in E$, there exists a quasiconformal homeomorphism $f : M \to M$ such that $f(E) = E$ and $f(x) = y$.

Following J. Gong and G. Martin, we focus on the following three notions. Let M be a Riemannian manifold of dimension $d \geq 2$ and $E \subset M$ be a closed subset; let $QC(M)$ denote the group of quasiconformal mappings of M and $QC(M, E)$, those elements which preserve E.

1. The set E is called *quasiconformally homogeneous relative to* M if $QC(M.E)$ acts transitively.
2. The set E is called *uniformly quasiconformally homogeneous relative to* M if we can choose K-quasiconformal maps from $QC(M, E)$ for each pair of points in E for some uniform K.
3. The set E is *strongly quasiconformally homogeneous relative to* M if there is a locally compact subgroup of $QC(M, E)$ acting transitively on E.

Finally, if we pick $E = M$, then we only speak of (uniformly or strong) homogeneity of M.

We have the following self-improving property based on the compactness of quasiconformal mappings.

Theorem 4.12. *Let E be a quasiconformally homogeneous compact subset of M, and assume that M is not homeomorphic to a sphere $\mathbb{S}^d$, $d \geq 3$. Then E is uniformly quasiconformally homogeneous.*

This was proved by P. MacManus, R. Näkki and B. Palka for compact subsets of $\widehat{\mathbb{C}}$ [MNP] and more generally by J. Gong and G. Martin [GonM].

In the Riemann sphere, we have the following classification.

Theorem 4.13 (MacManus, Näkki & Palka [MNP])**.** *Let E be a quasiconformally homogeneous compact subset of $\widehat{\mathbb{C}}$. Then E belongs to one of the following cases.*

- *$E = \widehat{\mathbb{C}}$;*
- *E is a finite set of points;*
- *E is the finite union of quasicircles that constitute the boundary components of a domain in $\widehat{\mathbb{C}}$;*
- *E is a Cantor set of Hausdorff dimension* $\dim_H E < 2$.

Moreover, J. Gong and G. Martin prove that strongly quasiconformally homogeneous compact subsets of the plane are also homogeneous by a uniform quasiconformal group. The question in higher dimension remains open [GonM].

Let us remark that if an annulus is quasiconformally homogeneous, it is not uniformly quasiconformally homogeneous: as points approach the boundary, the dilatation has to diverge.

Let us note that Theorem 4.2 implies that strongly quasiconformal homogeneous compact subsets are homeomorphic to Riemannian manifolds [GonM].

Concerning quasiconformal homogeneity of manifolds ($E = M$), we note that there are no obstructions. The situation is much different if we demand uniform homogeneity. We focus on hyperbolic manifolds.

Theorem 4.14 (Bonfert-Taylor, Canary, Martin & Taylor [BTCMT])**.** *Let M be a non simply connected and complete hyperbolic manifold of dimension $d \geq 2$. If M is uniformly quasiconformally homogeneous then M has homotopically nontrivial curves whose length is controlled explicitly by d and the uniform dilatation K, there are embedded hyperbolic balls in M whose diameter is bounded below in terms of d and K, and the limit set of $\pi_1(M)$ is the whole sphere.*

Moreover, a hyperbolic manifold of dimension $d \geq 3$ is uniformly quasiconformally homogeneous if and only if it is the regular cover of a compact manifold.

The situation between dimension 2 and higher is different. In dimension 2, there are examples of uniformly quasiconformally homogeneous Riemann surfaces which are not regular covers of closed surfaces, nor even deformations of regular covers [BTCST]. We have the following characterization of regular covers of compact Riemann surfaces.

Theorem 4.15 (Bonfert-Taylor, Martin, Reid & Taylor [BTMRT])**.** *An orientable hyperbolic surface is a regular cover of a closed hyperbolic orbifold if and only if there exists some K such that, for any pairs of points x, y, there exists a K-quasiconformal map sending x to y which is homotopic to a conformal homeomorphism.*

Remark 4.16. *Examples of homogeneous sets are constructed in [GonM]; see also the references therein. There are a series of results which consist in bounding from below the dilatation required to be uniformly homogeneous; see [BTCT].*

5 Actions on metric spaces

We consider compact metric spaces which admit a large group of uniform quasi-Möbius homeomorphisms and study their geometry. In particular, we point out properties which are preserved by quasi-Möbius mappings. Examples are provided by subRiemannian manifolds. Loewner spaces form an interesting class of metric spaces on which quasi-Möbius maps have strong properties. In particular, we discuss differentiability in the sense of Cheeger and show how to adapt Riemannian methods to this setting. These will have consequences on hyperbolic geometry. Combinatorial tools are also defined which are particularly useful in metric spaces. We first set up our framework by introducing the conformal gauge of a metric space.

5.1 The conformal gauge of a compact metric space

We let (X, d_X) be a compact metric space so that we may identify quasi-Möbius mappings with quasisymmetric maps. Following J. Heinonen [Hen, Chap. 15], we define the *conformal gauge* $\mathcal{G}(X)$ of the metric space X as the set of distances d on X such that the identity mapping $\mathrm{Id} : (X.d_X) \to (X, d)$ is quasi-Möbius (the name was suggested by D. Sullivan).

Given a metric $d \in \mathcal{G}(X)$, the operation of *snowflaking* d means that we consider powers d^α, $\alpha > 0$. As soon as $\alpha \in (0, 1)$, we obtain a genuine metric and this metric belongs to $\mathcal{G}(X)$ as well. Characterizing which metrics appear as such a power is the object of [TW].

Denote by $QM(X)$ the group of quasi-Möbius maps $h : X \to X$. Note that $QM(X)$ acts on $\mathcal{G}(X)$ by left translation $(h, d) \mapsto d_h$ defined by $d_h(x, y) = d(h^{-1}(x), h^{-1}(y))$ for $x, y \in X$. This setting provides some similarities with Teichmüller theory, which have not been systematically exploited yet.

In this paragraph, we are interested in three different kinds of questions:

1. Which properties on X hold for any metric from its gauge?
2. Is there a better metric in the gauge to work with?
3. How to recognize some classical metric spaces up to a quasi-Möbius change of metric?

For the first problem, besides purely topological properties which are automatically invariant, the following properties define invariants of a conformal gauge, see [Hen, Chap. 15].

Theorem 5.1. *Let X be a compact metric space. The following properties are quantitative quasisymmetric invariants.*

1. *The doubling property: there exists a number N_D such that any set can be covered by at most N_D sets of half its diameter.*
2. *Uniform perfectness: the diameter of any ball is comparable to its radius (as soon as the radius does not exceed the size of X).*
3. *Uniform disconnectedness: there exists $\varepsilon_0 > 0$ such that, for any pair of points $\{x, y\}$, there exists no chain $\{x_j\}_{0 \le j \le n}$ with $x_0 = x$, $x_n = y$ and $d(x_j, x_{j+1}) \le \varepsilon_0$.*
4. *The bounded turning property: there is a constant $T > 0$ such that any pair of points $\{y, z\}$ can be joined by a continuum K such that* $\mathrm{diam} K \le T d(y, z)$.
5. *Linear local connectedness: there is a constant $\lambda \ge 1$ such that, for any $x \in X$, $r \in (0, \mathrm{diam} X)$, the following two complementary properties hold:*
   - *for every $y, z \in B(x, r)$, there is a continuum $K \subset B(x, \lambda r)$ containing $\{y, z\}$;*
   - *for every $y, z \notin B(x, r)$, there is a continuum $K \in X \setminus B(x.r/\lambda)$ containing $\{y, z\}$.*
6. *Annular local connectedness: there is a constant $\lambda \ge 1$ such that, for any $x \in X$, $r \in (0, \mathrm{diam} X)$ and $y, z \in B(x, 2r) \setminus B(x, r)$, there a continuum $K \subset B(x, \lambda r)$ disjoint from $B(x, r/\lambda)$ containing $\{y, z\}$.*
7. *The doubling property of a fixed measure μ on X: there exists a constant $C > 0$ such that, for any ball B, $\mu(2B) \le C\mu(B)$ holds.*

Ahlfors regular conformal gauge.— A metric space X is *Ahlfors regular* if there is a Radon measure μ such that for any $x \in X$ and $r \in (0, \operatorname{diam} X]$, $\mu(B(x,r)) \asymp r^Q$ holds for some given $Q > 0$ [Mtl]. The measure μ is equivalent to the Hausdorff measure of X of dimension Q. Alhfors regularity is not invariant under quasisymmetric mappings; note that these measures are doubling —a robust property as claimed above. Given a metric space X, S. Semmes shows that there exists an Ahlfors regular metric in $\mathcal{G}(X)$ if and only if X is uniformly perfect and doubling [Hen, Thm 14.16].

The subset of Ahlfors regular metrics $\mathcal{G}_{AR}(X)$ in $\mathcal{G}(X)$ defines the *Ahlfors regular conformal gauge* of X. It has been described by M. Carrasco Piaggio in [CP1].

The *Ahlfors regular conformal dimension* $\operatorname{confdim}_{AR} X$ of X is defined as the infimum over $\mathcal{G}_{AR}(X)$ of every dimension Q [McT, CP1, Haï2]. This is a numerical invariant of the conformal gauge of X. It is a refinement due to M. Bourdon and H. Pajot [BP4] of P. Pansu's notion of conformal dimension $\operatorname{confdim} X$ defined as the infimum of the Hausdorff dimensions of (X,d) for $d \in \mathcal{G}(X)$ [Pan4]. The following properties hold for an Ahlfors regular metric space X:

$$\dim_{top} X \le \operatorname{confdim} X \le \operatorname{confdim}_{AR} X \le \dim_H X\,.$$

A difficult question is to determine whether these conformal dimensions can be attained by a metric from the gauge of X. Such a metric space should be the *nicest* in the sense that it is the less wrinkled (think of the Euclidean circle and its snowflakes —in particular the von Koch curve).

We will see later on that these numerical invariants capture very strong properties of the space and of its gauge. We now relate the existence of families of curves with positive modulus to conformal dimension.

Theorem 5.2. *Let (X, d_X) be a Q-regular metric space with $Q > 1$.*

1. *If there exists a family of curves Γ with positive Q-modulus $\operatorname{mod}_Q \Gamma > 0$ then*
$$\operatorname{confdim} X = \operatorname{confdim}_{AR} X = Q\,.$$
2. *If $\operatorname{confdim}_{AR} X = Q$, then X admits a tangent space which carries a family of positive Q-modulus.*

The first part of the theorem is due to J. Tyson [Tys] (see also [Hen, Thm 15.10]) and the converse is due to S. Keith and T. Laakso [KL]; see also [CP1, Cor. 1.5].

Characterizations of classical metric spaces. — We provide several statements which show the importance of the properties described above which guarantee the characterization of some low dimension spaces.

The first statement concerns the standard ternary Cantor set [DS, Prop. 15.11].

Theorem 5.3 (David & Semmes)**.** *The conformal gauge of a compact metric space X contains the ternary Cantor set if and only if X is uniformly perfect, uniformly disconnected and doubling.*

Euclidean circles have been characterized by P. Tukia and J. Väisälä [TV1], extending the planar characterization [Ahl].

Theorem 5.4 (Tukia & Vaisälä)**.** *The conformal gauge of a metric circle contains the Euclidean circle* $\mathbb{S}^1$ *if and only if it is doubling and satisfies the bounded turning property.*

The case of $\mathbb{S}^2$ has been dealt by M. Bonk and B. Kleiner [BnK1].

Theorem 5.5 (Bonk & Kleiner)**.** *A metric* 2*-sphere is quasisymmetrically equivalent to the Euclidean sphere* $\mathbb{S}^2$ *if and only if its conformal gauge is linearly locally connected and if it contains a* 2*-Ahlfors regular metric.*

For spheres of higher dimension, the general problem remains open. Theorem 5.13 provides a positive answer but requires a large group of quasi-Möbius mappings.

5.2 Loewner spaces

In the mid-nineties, J. Heinonen and P. Koskela introduced an important class of metric spaces which enabled to generalize the theory of quasiconformal mappings in a very satisfactory fashion [HnK2]; see also [Hen, HKST2]. In this sense, finding a Loewner structure in the gauge of a metric space is a good answer to question 2. above.

Let $Q > 1$; a metric space X is said to satisfy the *Q-Loewner property* if there exists a decreasing function ϕ such that, for any continua $\{E, F\} \subset X$,

$$\mathrm{cap}_Q(E, F) \geq \phi(\Delta(E, F))$$

where

$$\Delta(E, F) = \frac{\mathrm{dist}(E, F)}{\min\{\mathrm{diam}\, E.\mathrm{diam}\, F\}}$$

denotes the *relative distance* between E and F.

Loewner spaces contain a lot of rectifiable curves and behave, in many respect, as Riemannian manifolds with non-negative Ricci curvature. It follows from Theorem 5.2 that $Q = \mathrm{confdim} X$ if X is also Q-regular.

Remark 5.6. *The relative distance is well-behaved under quasi-Möbius mappings: given a distortion function θ, there is an increasing homeomorphism $\widehat{\theta} : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any θ-quasi-Möbius map $h : X \to Y$,*

$$\Delta(h(E), h(F)) \leq \widehat{\theta}(\Delta(E, F))$$

holds for any condenser (E, F) in X.

Let us note that when the space is also Q-regular, then one obtains very interesting bounds on capacities:

Proposition 5.7. *If X is a Q-regular metric space, then there is a decreasing homeomorphism $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\mathrm{cap}_Q(E,F) \le \psi(\Delta(E,F))$$

holds for any condensor.

In this setting, we may work with J. Ferrand's crossratio which behaves as the metric crossratio. These capacity estimates enable us to adapt Theorem 2.10 and Theorem 2.11 [HnK2, HKST1, BKR, BP2].

Theorem 5.8. *Let X be a Q-Loewner and Q-Ahlfors regular compact metric space, Y a linearly connected and Q-Ahlfors regular metric compact space and $f : X \to Y$ be a homeomorphism. The following proposition are equivalent:*

1. *f is a quasi-Möbius map;*
2. *f is an F-quasi-Möbius map;*
3. *f is quasiconformal;*
4. *f preserves the Q-moduli of curves up to a fixed factor.*

Moreover, if these conditions are satisfied, then f is absolutely continuous and absolutely continuous on Q-almost every curve. The space Y is also Q-Loewner.

In metric spaces, we define a quasiconformal map as a homeomorphism $f : X \to Y$ for which there is a constant H such that, for any $x \in X$,

$$H_f(x) = \limsup_{r\to 0} H_f(x,r) \le H$$

where

$$\begin{cases} H_f(x,r) = \dfrac{L_f(x,r)}{\ell_f(x,r)}, \\ \\ L_f(x,r) = \sup\{d_Y(f(x),f(z)),\ d_X(x,z)\le r\}, \\ \\ \ell_f(x,r) = \inf\{d_Y(f(x),f(z)),\ d_X(x,z)\ge r\}\,. \end{cases}$$

In particular, the above theorem shows that this local condition implies a global control. Other characterizations can be found in [BKR].

General Problem. — *Given a metric space, determine whether its gauge contains a Q-regular and Q-Loewner metric.*

For metric surfaces, a Q-Loewner Q-Ahlfors structure can only hold for $Q = 2$, see [BnK1] for spheres and more generally [HP, Cor. 3.14].

5.3 Geometric properties of metric spaces

The conformal elevator principle provides us with the following properties [BnK2, BnK4] which quantify the topological properties.

Theorem 5.9 (Bonk & Kleiner). *Let Z be compact connected metric space with a large group of quasi-Möbius mappings in the sense of Def. 3.10.*

1. *The space Z is doubling.*
2. *Any tangent space is quasi-Möbius equivalent to the complement in Z of a point.*
3. *The space Z is linearly locally connected.*

In particular $\mathcal{G}_{AR}(X) \neq \emptyset$. Together with Theorem 5.2 (and extra work!), this leads to

Corollary 5.10 (Bonk & Kleiner [BnK3]). *Let Z be compact connected metric space with a large group of quasi-Möbius mappings in the sense of Def. 3.10. If* $\mathrm{confdim}_{AR} X\,(>1)$ *is attained in* $\mathcal{G}_{AR}(X)$*, then there is a Loewner metric in the gauge.*

The main steps of its proof go roughly as follows. The assumption of the corollary, Theorem 5.2 and the selfsimilarity of the space shows that Z carries a family of curves of positive Q-modulus. Then the dynamics of the group and the quasi-invariance of moduli enable the authors to spread this family of curves everywhere and in every direction so that the Loewner condition follows.

Determining whether a compact metric space admits a Loewner metric in its gauge is a subtle issue. In particular, there is no known sufficient topological assumption on the compact space which ensures its existence. We record the following:

Theorem 5.11 (Bourdon & Pajot [Bou3, BP1, BP4]). *Let X be the Menger sponge.*

1. *There exist countably many different conformal gauges on X which admit a large group of quasi-Möbius maps and a Loewner metric.*
2. *There exist conformal gauges on X which admit a large group of quasi-Möbius maps but no Loewner metric.*

These examples correspond to boundaries of word hyperbolic groups, cf. § 6. The first family of examples also appear as the boundary of Fuchsian buildings.

One question is to determine which connected compact sets have conformal dimension 1. The following criteria were obtained by J. Mackay [Mac] and M. Carrasco Piaggio [CP2].

Theorem 5.12. *Let Z be a compact metric space with a large group of quasi-Möbius mappings.*

1. *If Z has no local cut points, then* $\mathrm{confdim} Z > 1$.

2. *If there is a sequence $(\delta_n)_n$ tending to zero and a sequence of finite sets $P_n \subset Z$ such that, for any component Y of $Z \setminus P_n$,* $\mathrm{diam}\, Y \leq \delta_n$, *then* $\mathrm{confdim}_{AR} Z = 1$.

There remains a gap between these two statements: if there is a dense collection of local cut points but which do not satisfies the condition **2.**, then it is still unknown whether the conformal dimension can be one or not.

We mention a characterization of Euclidean spheres $\mathbb{S}^n$ which assumes a large group of quasi-Möbius maps.

Theorem 5.13 (Bonk & Kleiner [BnK2])**.** *A compact metric space Z is quasi-Möbius equivalent to a Euclidean sphere $\mathbb{S}^n$, $n \geq 1$, if and only if the following three conditions hold:*

1. *the space Z admits a large group G of uniformly quasi-Möbius maps;*

2. *there is a metric minimizing* $\mathrm{confdim}_{AR}(Z)$*;*

3. *the identity* $\mathrm{confdim}_{AR} X = \dim_{top} X$ *holds.*

Moreover, G is conjugate to a group of Möbius transformations.

Let us remark that the conclusion includes the fact that Z is homeomorphic to a sphere. The main idea of the proof is to establish that Z admits a tangent bi-Lipschitz equivalent to some $\mathbb{R}^n$. This implies that Z is quasi-Möbius equivalent to $\mathbb{S}^n$. Theorem 4.6 shows that G is conjugate to a group of Möbius mappings. Finding the proper tangent space is the most delicate issue: the dimension assumption provides us with a Lipschitz map $f : Z \to \mathbb{S}^n$, for $n = \dim_{top} Z$, such that the image of f has positive Lebesgue measure. Two consecutive zooming arguments enable us to first obtain a Lipschitz map of bounded multiplicity between their respective tangent spaces, and then a new tangent space bi-Lipschitz to $\mathbb{R}^n$; a purely topological argument implies that a Lipschitz map of bounded multiplicity onto $\mathbb{R}^n$ is locally bi-Lipschitz somewhere.

We close this section with a somewhat surprising example [KK, § 7].

Theorem 5.14 (Kapovich & Kleiner)**.** *There exist compact sets with the following properties.*

- *The group of homeomorphisms is a discrete uniform convergence group.*

- *There exists a unique (Ahlfors-regular) conformal gauge which admits a large group of quasi-Möbius maps. For this gauge, the group of homeomorphisms is uniformly quasi-Möbius.*

The construction is quite involved. These are two-dimensional compact spaces made of infinitely many 2-spheres glued along circles following an intricate pattern. Such examples cannot exist in smaller dimension.

5.4 Combinatorial moduli and the combinatorial Loewner property

In the realm of (quasi)conformal geometry, discretization methods have proved to be very powerful. The basic idea is to approximate a metric space by a collection of balls since conformal maps have the property of respecting their geometry. Such tools play an important role in proving properties of quasi-Möbius mappings, especially in Ahlfors regular Loewner spaces, cf. [Pan4, HnK1, Can2, Tys, BnK1, BdK1] among others.

Let $\mathcal{S}$ be a covering of a topological space X, and let $p \geq 1$. Denote by $\mathcal{M}_p(\mathcal{S})$ the set of functions $\rho : \mathcal{S} \to \mathbb{R}_+$ such that $0 < \sum \rho(s)^p < \infty$; elements of $\mathcal{M}_p(\mathcal{S})$ we call *admissible metrics*. For $K \subset X$ we denote by $\mathcal{S}(K)$ the set of elements of $\mathcal{S}$ which intersect K. The *ρ-length* of K is by definition

$$\ell_\rho(K) = \sum_{s \in \mathcal{S}(K)} \rho(s)\,.$$

Define the *ρ-volume* by

$$V_p(\rho) = \sum_{s \in \mathcal{S}} \rho(s)^p\,.$$

If Γ is a family of curves in X and if $\rho \in \mathcal{M}_p(\mathcal{S})$, we define

$$L_\rho(\Gamma, \mathcal{S}) = \inf_{\gamma \in \Gamma} \ell_\rho(\gamma),$$

$$\mathrm{mod}_p(\Gamma, \rho, \mathcal{S}) = \frac{V_p(\rho)}{L_\rho(\Gamma, \mathcal{S})^p},$$

and the *combinatorial modulus* by

$$\mathrm{mod}_p(\Gamma, \mathcal{S}) = \inf_{\rho \in \mathcal{M}_p(\mathcal{S})} \mathrm{mod}_p(\Gamma, \rho, \mathcal{S}).$$

Note that if $\mathcal{S}$ is a finite cover, then the modulus of a nonempty family of curves is always finite and positive.

Under suitable conditions, the combinatorial moduli obtained from a sequence $(\mathcal{S}_n)_{n\geq 1}$ of coverings can be used to approximate analytic moduli. One such condition requires the sequence to be a *uniform family of quasipackings*; compare with [Bou7] for an equivalent point of view.

Definition 5.15 (Quasipacking)**.** *A* quasipacking *of a metric space is a locally finite cover $\mathcal{S}$ such that there is some constant $K \geq 1$ which satisfies the following property. For any $s \in \mathcal{S}$, there are two balls $B(x_s, r_s) \subset s \subset B(x_s, K \cdot r_s)$ such that the family $\{B(x_s, r_s)\}_{s \in \mathcal{S}}$ consists of pairwise disjoint balls. A sequence $(\mathcal{S}_n)_{n\geq 1}$ of quasipackings is called* uniform *if the mesh of $\mathcal{S}_n$ tends to zero as $n \to \infty$ and the constant K defined above can be chosen independent of n.*

For compact Ahlfors regular spaces, uniform sequences of finite quasipackings always exist, and are preserved under quasisymmetric maps, quantitatively. It is sometimes convenient to start from a uniform sequence of quasipackings $(\mathcal{S}_n)_n$ such that, at each level n, $\operatorname{diam} s \asymp \delta^n$ holds for some $\delta \in (0,1)$ and any $s \in \mathcal{S}_n$. In that case, we call $(\mathcal{S}_n)_n$ an *approximation of* X. Note that this notion appeared implicitely in § 3.2 when defining the Hausdorff-Gromov convergence.

The next result says that under appropriate hypotheses, analytic and combinatorial moduli are comparable [Haï1, Prop. B.2].

Proposition 5.16. *Suppose $Q > 1$, X is an Ahlfors Q-regular compact metric space, and $(\mathcal{S}_n)_{n\geq 1}$ is a sequence of uniform quasipackings. Let Γ be a nondegenerate closed family of curves in X. Then either*

1. $\operatorname{mod}_Q\Gamma = 0$ *and* $\lim_{n\to\infty} \operatorname{mod}_Q(\Gamma, \mathcal{S}_n) = 0$, *or*
2. $\operatorname{mod}_Q\Gamma > 0$, *and there exist constants $C \geq 1$ and $N \in \mathbb{N}$ such that for any $n > N$,*
$$\frac{1}{C}\operatorname{mod}_Q(\Gamma, \mathcal{S}_n) \leq \operatorname{mod}_Q\Gamma \leq C\operatorname{mod}_Q(\Gamma, \mathcal{S}_n).$$

We deduce the following general criteria on the Ahlfors-regular conformal dimension.

Corollary 5.17. *Let X be a Q-Ahlfors-regular compact metric space, and $(\mathcal{S}_n)_{n\geq 1}$ a sequence of uniform quasipackings. If $Q > \dim_{AR} X$, then*

$$\lim_{n\to\infty} \operatorname{mod}_Q(\Gamma, \mathcal{S}_n) = 0$$

for any family of curves the diameter of which have a positive lower bound.

The following is based on the length-area method [Pan4].

Proposition 5.18 (Pansu)**.** *Let X be a Q-Ahlfors-regular metric space, and $(\mathcal{S}_n)_{n\geq 1}$ is a sequence of uniform quasipackings. Assume there exists a family of curves Γ the diameter of which has a positive lower bound and a probability measure μ on Γ such that $\mu(\Gamma(s)) \lesssim (\operatorname{diam} s)^{Q-1}$ for all s, where $\Gamma(s)$ denotes the subfamily of curves of Γ which go through s. Then $\operatorname{confdim}_{AR}X = Q$.*

Assuming that X has a large group of quasi-Möbius maps with a linear distortion function, we have the following characterization, which has the advantage of not starting from a candidate metric:

Theorem 5.19 (Keith & Kleiner, Carrasco Piaggio [CP1])**.** *Let X be an Ahlfors-regular connected metric space, and $(\mathcal{S}_n)_{n\geq 1}$ an approximation of X. Assume that X admits a large group of quasi-Möbius maps controlled by a uniform linear distortion function. Then there exists $\delta > 0$ such*

$$\operatorname{confdim}_{AR}X = \inf\{Q \geq 1,\ \lim \operatorname{mod}_Q(\Gamma_\delta, \mathcal{S}_n) = 0\}$$

where Γ_δ denotes the family of curves of diameter at least δ.

The linear distortion assumption ensures that the quasi-Möbius maps are all bi-Lipschitz (with no uniform bound).

Remark 5.20. *Combinatorial moduli are used to prove that quasi-Möbius maps preserve Q-modulus in Q-Ahlfors regular metric spaces, and for the proofs of Theorem 5.5 and Theorem 5.12 2. among others.*

The combinatorial Loewner property. — This notion was introduced by B. Kleiner to capture combinatorially the properties of Q-Loewner and Q-Ahlfors regular metric spaces [Klr]; see [BdK1] for a systematic study of this notion. Let $Q > 1$ and $(\mathcal{S}_n)$ be an approximation of X; a metric space X is said to satisfy the *combinatorial Q-Loewner property* if there exist decreasing functions ϕ, ψ such that, for any continua $\{E, F\} \subset X$, there exists n_0 such that, if $n \geq n_0$, then

$$\phi(\Delta(E,F)) \leq \mathrm{cap}_Q(E,F,\mathcal{S}_n) \leq \psi(\Delta(E,F))$$

where we recall that

$$\Delta(E,F) = \frac{\mathrm{dist}(E,F)}{\min\{\mathrm{diam}\, E . \mathrm{diam}\, F\}}$$

denotes the relative distance between E and F.

The following theorem justifies its interest.

Theorem 5.21 (Bourdon & Kleiner [Klr, BdK1])**.** *We have the following properties.*

1. *A Q-Loewner Q-regular metric space with* $Q > 1$ *satisfies the combinatorial Q-Loewner property.*
2. *The combinatorial Loewner property is invariant under quasi-Möbius mappings.*

The converse is the main issue for this notion but remains an open question:

Conjecture 5.22 (Kleiner [Klr])**.** *An Ahlfors regular metric space which satisfies the combinatorial Q-Loewner property contains a Q-Loewner Q-regular metric in its conformal gauge.*

Examples of spaces which satisfy the combinatorial Loewner property include the standard Sierpiński carpet and Menger sponge, as well as some boundaries of word hyperbolic Coxeter groups [BdK1]. Other examples have been studied by A. Clais [Cli].

As a concluding remark, let us mention that moduli of curves play a central role in the understanding of quasiconformal geometry. Unfortunately, Theorem 5.2 tells us that they can only be used when the dimension of the space is minimal within its gauge. On the contrary, combinatorial moduli can be considered in any metric space. Since uniform packings behave well under quasi-Möbius maps, it can be hoped that the combinatorial Loewner property can be established more easily: it does not depend on the metric we are starting with. Thus, a positive answer to the conjecture should help recognizing conformal gauges which contain Loewner structures.

5.5 SubRiemannian manifolds

Let M be a compact manifold and let $H \subset TM$ be a subbundle. For each local frame $(X_1, \ldots, X_h)$ of H at $x \in M$ and $i \geq 1$, let H_x^i denote the subspace of T_xM spanned by $X_1, \ldots, X_h$ together with all commutators of these vector fields of order at most i. The subbundle H is called *equiregular* if for all $x \in M$, $\dim H_x^i$ is independent of x, and *horizontal* if $H^n = TM$ for some n.

We call the pair $(M; H)$ a *Carnot-Carathéodory space* of depth n (abbreviated "cc space") if H is equiregular and horizontal, and $n = \inf\{k; H_x^k = T_xM\}$. To each subspace H_x, we consider an inner product g_x which depends continuously on x. The triple (M, H, g) gives rise to a *subRiemannian manifold.*

A piecewise smooth curve $\gamma : [0, 1] \to M$ is *horizontal* if γ' is contained in H almost everywhere. We may then define the *Carnot-Carathéodory metric* d_{cc} on M as

$$d_{cc}(x, y) = \inf \ell(\gamma)$$

where the infimum is taken over all horizontal curves joining x and y. According to Chow, this defines a metric on M. If instead of an inner metric, we just consider a continuous collections of norms, then one obtains a subFinsler structure.

Let (M, H, g) be an equiregular Carnot-Carathéodory space of depth n. Then it is Q-Ahlfors regular [Mit] for

$$Q = \sum_{k=1}^{n} k(\dim H^k - \dim H^{k-1})\,.$$

These provide examples of Loewner spaces [HnK2].

5.5.1 Carnot groups

A *Carnot group* G is a simply connected nilpotent Lie group, together with a derivation α of its Lie algebra $\mathfrak{g}$ such that the subspace V_1 with eigenvalue 1 is horizontal (and equiregular) and is endowed with an inner product. The derivation gives rise to a one-parameter dilation group on the Lie group [Pan5]. More precisely, the Lie algebra $\mathfrak{g}$ splits into

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \ldots \oplus V_s$$

where $V_{j+1} = [V_1, V_j]$ for $j = 1, \ldots, s-1$ and $[V_1, V_s] = \{0\}$; the grading introduced for subRiemannian manifolds corresponds to $H_k = V_1 \oplus \cdots \oplus V_k$. The dilations on G come from $e^{\alpha t}(v) = t^j v$ for $v \in V_j$ via the exponential map (which is a diffeomorphism). Examples are provided in § 5.5.3. Carnot groups are the simplest examples of Carnot-Carathéodory manifolds and serve as infinitesimal models for them:

Theorem 5.23 (Mitchell [Mit])**.** *Weak tangent spaces of equiregular Carnot-Carathéodory spaces are Carnot groups.*

We close this section with the following characterization of Carnot groups based on the solution to the fifth Hilbert problem:

Theorem 5.24 (Le Donne [LD])**.** *The subFinsler Carnot groups are the only metric spaces (X, d) that are:*

1. *locally compact and geodesic;*
2. *isometrically homogeneous: for any $p, q \in X$, there is a distance-preserving homeomorphism of X mapping p to q;*
3. *self-similar: there exists $\lambda > 1$ and a homeomorphism f of X such that $d(f(p), f(q)) = \lambda d(p, q)$ for all $p, q \in X$.*

5.5.2 Differentiability

The theory of quasi-Möbius mappings and quasiconformal maps acting on sub-Riemannian manifolds is well developed, and has been a source of inspiration for the notion of Loewner spaces, see for instance [Pan5, Rei, MM]. In particular, one can give sense to the differential of a quasi-Möbius map between Carnot groups: Let δ_t be the one parameter group of dilations at the origin $0 \in G$, and δ'_t for $0 \in G'$. Following P. Pansu, a map $f : G \to G'$ is differentiable at the origin if $\lim_{t\to 0}(\delta'_t)^{-1} \circ f \circ \delta_t$ is convergent towards a group homomorphism which commutes with the derivations. In the setting of cc-manifolds, one considers quasi-Möbius maps $f : M \to M'$ and say that f is differentiable at x if

$$\lim_{t\to\infty} \left\{(M, td_{cc}) \xrightarrow{f} (M', td_{cc})\right\}$$

is convergent to a group homomorphism between their respective tangent Carnot groups.

Theorem 5.25. *A quasi-Möbius map between cc-spaces is differentiable almost everywhere and the differential is a group automorphism which is compatible with the derivation. Moreover, 1-quasiconformal maps on Carnot groups are compositions of dilations and isometries.*

The theorem was proved by P. Pansu in the setting of Carnot groups [Pan5] and by G. Margulis and G. Mostow in the general setting [MM].

5.5.3 Heisenberg groups

Let $\mathbb{K}$ denote either the field of complex or quaternionic numbers, or the division algebra of Cayley numbers. Fix $n \geq 2$ (and $n = 2$ in the case of the Cayley algebra) and let us consider the simply connected Lie group $G^n_{\mathbb{K}} = \mathbb{K}^{n-1} \times \Im m\mathbb{K}$ with multiplication

$$(z, t) \cdot (w, s) = (z + w, t + s + \Im m\langle z, w\rangle)\,.$$

The Lie algebra splits as follows: $\mathfrak{g} = H_n \oplus V$ where $H_n = \mathbb{K}^{n-1}$ and $V = \Im m\mathbb{K}$. We endow this with the derivation given in the above decomposition by

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

so that $G^n_{\mathbb{K}}$ becomes a Carnot group of depth 2 and with horizontal space H_n.

The following theorem extends Theorem 4.6 (with a similar proof):

Theorem 5.26 (Chow [Cho]). *A countable group of uniform quasi-Möbius homeomorphisms of the Alexandrov compactification of $G^n_{\mathbb{C}}$, $n \geq 2$, is quasisymmetrically conjugate to a group of conformal transformations provided the set of conical points has positive measure.*

The other situation is rigid so that any group of quasi-Möbius maps is automatically a group of conformal transformations:

Theorem 5.27 (Pansu [Pan5]). *A quasi-Möbius map on the Alexandrov compactification of $G^n_{\mathbb{K}}$ is the composition of an inversion, a dilation and an isometry when $\mathbb{K}$ is the field of quaternionic numbers or the division algebra of Cayley numbers.*

P. Pansu proves that, in this setting, the group of automorphisms which preserves the Carnot structure only contains similarities. Since quasi-Möbius are differentiable almost everywhere, it follows that a quasi-Möbius map is always 1-quasiconformal, hence conformal according to Theorem 5.25.

We will see applications of these results to the classification of hyperbolic manifolds of negative curvature.

5.6 Cheeger differentiability

We introduce the notion of differentability in the sense of Cheeger [Che]; see also [Kei, KrM, Bat, CKS] and [HKST2] for a general account. The existence of a measurable differentiable structure is known to hold for complete Ahlfors regular Loewner spaces This notion will enable us to generalize differential calculus to metric spaces and adapt previous results on Riemannian manifolds to Loewner spaces. Moreover we will see that quasi-Möbius mappings become differentiable almost everywhere, see below.

By a metric measure space (X, d, μ), we will mean a metric space (X, d) endowed with a Borel regular measure μ which gives positive and finite mass to any non-empty open ball.

Definition 5.28 (measurable differentiable structure). *A measurable differentiable structure on a metric measure space (X, d, μ) is a countable collection of pairs $\{(X_\alpha, \mathbf{x}_\alpha)\}$ called coordinate patches, that satisfy the following conditions.*

1. *Each X_α is a measurable subset of X with positive measure, and the union $\cup_\alpha X_\alpha$ has full measure in X.*

2. *For each α, there is some $N(\alpha) \in \mathbb{N}$ such that $\mathbf{x}_\alpha : X \to \mathbb{R}^{N(\alpha)}$ is Lipschitz and, for any $\lambda \in \mathrm{Hom}(\mathbb{R}^{N(\alpha)}, \mathbb{R})$, $\lambda \circ \mathbf{x}_\alpha \equiv 0$ if and only if $\lambda = 0$. The dimensions $N(\alpha)$ are bounded independently of α and their maximum is called the dimension of the differentiable structure.*

3. *For each α, $\mathbf{x}_\alpha = (x_\alpha^1, \ldots, x_\alpha^{N(\alpha)})$ spans the differentials almost everywhere for X_α in the following sense: for every Lipschitz function $f : X \to \mathbb{R}$, there exists a measurable function $df^\alpha : X_\alpha \to \mathbb{R}^{N(\alpha)}$ so that for μ-a.e. $x \in X_\alpha$*

$$\limsup_{y\to x} \frac{|f(y) - f(x) - df^\alpha(x)(\mathbf{x}_\alpha(y) - \mathbf{x}_\alpha(x))|}{d(x,y)} = 0\,.$$

Moreover, df^α is unique up to sets of measure zero.

A *differentiabillity space* is a metric measure space endowed with a measurable differentiable structure.

It follows from the theory that, for each α and μ-a.e. $x \in X_\alpha$, $\mathrm{Hom}(\mathbb{R}^{N(\alpha)}, \mathbb{R})$ can be supplied with a norm defined for any linear form λ by

$$|\lambda|_x = \limsup_{y\to x} \frac{|\lambda \circ \mathbf{x}_\alpha(y) - \lambda \circ \mathbf{x}_\alpha(x)|}{d(x,y)}\,.$$

In particular, if f is a Lipschitz function, then, for almost every $x \in X_\alpha$,

$$|df^\alpha(x)|_x = \mathrm{Lip} f(x) = \limsup_{r\to 0} \sup_{d(x,y)\le r} \frac{|f(x) - f(y)|}{d(x,y)}\,.$$

This Banach space $T_x^*X = (\mathbb{R}^{N(\alpha)}, |\cdot|_x)$ defines the cotangent space at x, and they combine to give a vector bundle T^*X which is called the generalized cotangent bundle of X. By using local sections $d^\alpha f$ on X_α, one can define a derivation operator d on the algebra of locally Lipschitz functions on X which takes values in the bundle $\Gamma(T^*X)$ of sections in T^*X. A tangent bundle TX can then be defined by duality.

Theorem 5.29 (Heinonen, Koskela, Shanmugalingam & Tyson [HKST1, Thm. 10.8])**.** *Let (X, d_X, μ) and (Y, d_Y, ν) be two Q-Loewner Q-Ahlfors regular spaces for some $Q > 1$ endowed with measurable differentiable structures and let $h : X \to Y$ be a quasi-Möbius mapping. Then there is a natural induced map $h^* : T^*Y \to T^*X$ such that*

$$h^*(df) = d(f \circ h)$$

for all Lipschitz functions $f : Y \to \mathbb{R}$.

Let us remark that quasi-Möbius mappings being invertible the dimensions of the cotangent bundle have to preserved by the group.

5.7 The group of conformal maps

Let (X, d_X, μ) and (Y, d_Y, ν) be two Q-Loewner Q-Ahlfors regular spaces for some $Q > 1$, and suppose we are given measurable Riemannian structures $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_Y$ on T^*X and T^*Y respectively that are uniformly equivalent to the norms. Such structures exist since the cotangent bundles have finite dimension and we may consider the structure defined by the ellipsoid of largest volume contained in the unit ball of each cotangent space. Following B. Kleiner [Klr], a homeomorphism

$h : X \to Y$ is *conformal* with respect to these structures if it is quasiconformal and its derivative $d_x h : T^*_{h(x)}Y \to T^*_x X$ is conformal for almost every $x \in X$. In this case, it follows that the metric infinitesimal distortion H_h of h satisfies $H_h(x) = 1$ almost everywhere and that moduli of curves are preserved [BP2].

The *conformal group* of $(X, \langle\cdot,\cdot\rangle_X)$, denoted $\mathrm{Conf}(X, \langle\cdot,\cdot\rangle_X)$ is the group of conformal homeomorphisms.

If the dynamics of $\mathrm{Conf}(X, \langle\cdot,\cdot\rangle_X)$ is sufficiently rich, then there is a unique action up to conformality. This extends Mostow's rigidity theorem.

Theorem 5.30 (Kleiner [Klr, Thm 5.6])**.** *Let X, X' be Q-Loewner Q-regular metric spaces for some $Q > 1$ and assume that $\rho : G \to \mathrm{Conf}(X, \langle\cdot,\cdot\rangle)$ and $\rho' : G \to \mathrm{Conf}(X', \langle\cdot,\cdot\rangle')$ are two representations such that their respective actions are uniform convergence actions. Then ρ and ρ' are conformally equivalent.*

Let us note that the assumptions of this theorem hold when the cotangent bundle is one-dimensional (this happens for boundaries of Fuchsian buildings, cf. [BP2, Xie1]). In that case, quasi-Möbius maps are automatically conformal, so one obtains with Theorem 5.8 the following self-improving property.

Theorem 5.31 (Kleiner, cf. [Xie1, Thm.4.6])**.** *Let X be a Q-Loewner Q-regular compact metric space for some $Q > 1$ and assume that T^*X is one-dimensional. Then quasi-Möbius maps are uniformly quasi-Möbius.*

A part of the proof of Theorem 4.6 adapts to give:

Theorem 5.32. *Let G be a group of uniformly quasi-Möbius mappings on a Q-Loewner Q-regular metric space $(X, \langle\cdot,\cdot,\rangle_X)$. Then T^*X carries a measurable conformal structure invariant under G.*

6 Quasi-Möbius maps and hyperbolicity

Background on hyperbolic metric spaces include [Gro3, CDP, GdlH, KB]. After recalling the basic definitions and properties of hyperbolic spaces in the sense of M. Gromov, we relate their geometric properties to the quasiconformal geometry of their boundaries at infinity. We then use this correspondence to exhibit rigidity phenomena of hyperbolic manifolds, see also [Bou7].

Let X be a metric space. It is *geodesic* if any pair of points $\{x, y\}$ can be joined by a (geodesic) segment i.e, a map $\gamma : [0, d(x, y)] \to X$ such that $\gamma(0) = x$, $\gamma(d(x, y)) = y$ and $d(\gamma(s), \gamma(t)) = |t - s|$ for all $s, t \in [0, d(x, y)]$. The metric space X is *proper* if closed balls of finite radius are compact.

A *triangle* Δ in a metric space X is given by three points $\{x, y, z\}$ and three segments joining them two by two. Given a constant $\delta \geq 0$, the triangle Δ is δ-*thin* if any side of the triangle is contained in the δ-neighborhood of the two others.

Definition 6.1 (Hyperbolic spaces and groups)**.** *A geodesic metric space is hyperbolic if there exists $\delta \geq 0$ such that every triangle is δ-thin. A group G is word hyperbolic if it acts geometrically on a proper, geodesic hyperbolic metric space.*

Basic examples of hyperbolic spaces are the simply connected Riemannian manifolds $\mathbb{H}^n$ of sectional curvature (-1) and $\mathbb{R}$-trees. In particular fundamenal groups of closed hyperbolic manifolds are word hyperbolic.

A quasi-isometry between metric spaces X and Y is a map $\varphi : X \to Y$ such that there are constants $\lambda \geq 1$ and $c > 0$ such that:

- (quasi-isometric embedding) for all $x, x' \in X$, the two inequalities
$$\frac{1}{\lambda} d_X(x, x') - c \leq d_Y(\varphi(x), \varphi(x')) \leq \lambda d_X(x, y) + c$$
hold and
- the c-neighborhood of the image $f(X)$ covers Y.

This defines in fact an equivalence relation on metric spaces. Note that any two locally finite Cayley graphs of the same group are quasi-isometric: this enables us to discuss the quasi-isometry class of a finitely generated group (through the class of its locally finite Cayley graphs). More generally, Švarc-Milnor's lemma asserts that there is only one geometric action of a group on a proper geodesic metric space up to quasi-isometry [GdlH, Prop. 3.19].

Let $\eta, \eta' : (\mathbb{R}_+, 0) \to (\mathbb{R}_+, 0)$ be two homeomorphisms. We say that a map $f : X \to Y$ is (η, η')-biuniform if
$$\eta(|x - x'|) \leq |f(x) - f(x')| \leq \eta'(|x - x'|)$$
holds for all $x, x' \in X$.

Lemma 6.2 (Tukia). *Let $f : X \to Y$ be (η, η')-bi-uniform and suppose that X and $f(X)$ are geodesic spaces. Given $c > 0$, f is (λ, c)-quasi-isometric for λ depending only on c, η and η'.*

6.1 Basic properties

We briefly review some properties of hyperbolic geodesic spaces.

Approximation by trees.— Let $k \geq 1$ and Z be the union of k segments or rays with vertex w. There is a $(1, c)$-quasi-isometry of Z into a tree T where c only depends on δ and k.

Shadowing lemma.— A *quasigeodesic* is the image of an interval by a quasi-isometric embedding. The shadowing lemma asserts that, given δ, λ and c, there is a constant $H = H(\delta, \lambda, c)$ such that, for any (λ, c)-quasigeodesic q in a proper geodesic δ-hyperbolic metric space X, there is a geodesic γ at Hausdorff distance at most H.

It follows from the shadowing lemma that, among geodesic metric spaces, hyperbolicity is invariant under quasi-isometry : if X, Y are two quasi-isometric geodesic metric spaces, then X is hyperbolic if and only if Y is hyperbolic.

Compactification.— A proper geodesic hyperbolic space X admits a canonical compactification $X \sqcup \partial X$ at infinity in a similar spirit as the visual boundary introduced by P. Eberlein and B. O'Neill for visibility manifolds [EO]. This compactification can be defined by looking at the set of rays i.e., isometric embeddings $r : \mathbb{R}_+ \to X$, up to bounded Hausdorff distance. The topology is induced by the uniform convergence on compact subsets of $\mathbb{R}_+$. The boundary can be endowed with a family of *visual distances* d_v compatible with its topology i.e., which satisfy

$$d_v(a,b) \asymp e^{-\varepsilon d(w,(a,b))}$$

where $w \in X$ is any choice of a base point, $\varepsilon > 0$ is a *visual parameter* and (a,b) is any geodesic asymptotic to rays defining a and b. Visual distances are known to exist for visual parameters $\varepsilon > 0$ chosen small enough with respect to δ. When X is the hyperbolic space $\mathbb{H}^n$, $n \geq 1$, then $\partial\mathbb{H}^n$ may be endowed with a visual metric so that it is isometric to the Euclidean sphere $\mathbb{S}^{n-1}$.

If $\Phi : X \to Y$ is a quasi-isometry between two geodesic hyperbolic spaces, then the shadowing lemma implies that Φ induces a homeomorphism $\phi : \partial X \to \partial Y$. This means that a word hyperbolic group G admits a topological boundary ∂G defined by considering the boundary of any proper geodesic metric space on which G acts geometrically.

In the case of the fundamental group of a closed hyperbolic manifold of dimension, the boundary is homeomorphic to the $(n-1)$-dimensional sphere.

6.2 Analytic aspects

A general principle asserts that a geodesic hyperbolic group is determined by its boundary. More precisely, F. Paulin proved that the quasi-isometry class of a word hyperbolic group is determined by its boundary equipped with its quasiconformal structure [Pau]. This was later generalized by M. Bonk and O. Schramm to a broader context [BSm].

Quasi-isometries provide natural examples of quasi-Möbius maps:

Theorem 6.3. *A (λ, c)-quasi-isometry between proper, geodesic, metric spaces extends as a θ-quasi-Möbius map between their boundaries, where θ only depends on λ, c, the hyperbolicity constants and the visual parameters.*

This result takes its roots in the work of V.A. Efremovich and E.S. Tihomirova [ET]; see also [Mar] where quasi-isometries are explicitly defined and where Theorem 6.3 is proved for real hyperbolic spaces.

A pointed geodesic metric space (X, w) is *quasi-starlike* if there is some constant K such that any point $x \in X$ lies at distance at most K of a ray emanating from w. M. Bonk and O. Schramm's result reads

Theorem 6.4 (Bonk & Schramm [BSm])**.** *Two proper quasi-starlike geodesic hyperbolic metric spaces are quasi-isometric if and only if there is a quasi-Möbius homeomorphism between their boundaries.*

As a byproduct, one obtains:

Theorem 6.5 (Paulin [Pau]). *Two non-elementary word hyperbolic groups are quasi-isometric if and only if there is a quasi-Möbius homeomorphism between their boundaries.*

The latter enables us to define the (Ahlfors-regular) conformal gauge $\mathcal{G}_{(AR)}(G)$ of a word hyperbolic group G: it corresponds to the conformal gauge of its boundary supplied with any visual distance.

6.3 Groups of isometries versus quasi-Möbius actions

Theorem 6.3 has the following consequences.

Corollary 6.6. *Let G be a group of isometries acting on a proper geodesic hyperbolic space X. Let us endow ∂X with a visual distance. Then the action of G extends to a uniformly quasi-Möbius convergence action of ∂X. If the action of G on X is properly discontinuous, then its action on ∂X is a discrete action.*

Let G be a collection of (λ, c)-quasi-isometries on a proper geodesic hyperbolic metric space. We say that G is an *approximate group* if the following two properties hold:

- for any $f, g \in G$, there exist $h \in G$ and M such that
$$d((g \circ f)(x), h(x)) \leq M$$
holds for all $x \in X$;
- for any $f \in G$, there exist $h \in G$ and M such that
$$d((h \circ f)(x), x) \leq M \quad \text{and} \quad d((f \circ h)(x), x) \leq M$$
hold for all $x \in X$.

Corollary 6.7. *An approximate group G of (λ, c)-quasi-isometries of a proper geodesic hyperbolic space X extends to a uniformly quasi-Möbius convergence action of ∂X whenever ∂X is endowed with a visual distance.*

If X is quasi-starlike, then Theorem 6.4 implies the converse:

Theorem 6.8. *Let X be a hyperbolic, proper, geodesic and quasi-starlike metric space. Let us endow ∂X with a visual distance. If G is a group of uniform quasi-Möbius maps, then it extends as an approximate group of uniform quasi-isometries.*

One can prove that two quasi-isometries which share the same boundary values have bounded uniform distance. Thus, one usually considers the group of quasi-isometries up to bounded distance. Therefore, the previous results estabishes a correspondence between quasi-Möbius maps on the boundary and quasi-isometry classes.

We close this section with a stronger version of Theorem 3.13 —corresponding to the actual theorem, which provides a partial answer to the following question: when is a convergence group a subgroup of isometries of a hyperbolic metric space?

Theorem 6.9 (Bowditch, Yaman, Gerasimov)**.** *Let G be a countable group admitting a discrete convergence action on a metrizable compact space Z. If its diagonal action on the set of distinct pairs is compact, then G acts properly discontinuously on a proper geodesic hyperbolic metric space X by isometries and there is an equivariant homeomorphism $h : \partial X \to Z$.*

The general case is still open. There are essentially two different approaches for this theorem: first building a quasiconformal structure on Z and then show that it is the boundary of a hyperbolic space, or defining an unbounded metric space with a compactification homeomorphic to Z and use this space to define a metric structure on Z; see also the next section for similar ideas.

B. Bowditch's approach is to define a notion of crossratio invariant by the group, and to use this notion to define a metric structure on the set of triples. Showing that this structure is hyperbolic is the concluding step. The other approach, used by V. Gerasimov and L. Potyagailo, consists in constructing an unbounded metric space from the uniform structure of Z with an action of the group and in relating its Floyd boundary to Z (a substitute for the visual boundary). The metric structure coming from the Floyd boundary is used to establish the hyperbolicity of the space. [Ger1, Ger2, GerP].

6.4 Hyperbolic fillings: the space of snapshots

In previous sections, we have seen that the boundary at infinity of a (proper) geodesic hyperbolic space was a (compact) metric space endowed with a canonical conformal gauge. Conversely, given a compact metric Z, we may construct a hyperbolic space X the boundary of which is homeomorphic to Z with the same conformal gauge. This point of view has proved to have important impact on understanding the analytic properties of compact metric spaces. We will see examples here and in Section 7.

Let Z be a compact metric space of diameter D. We use the notation from § 3.2. For a ball $B \in \mathcal{B}_k$, we write $|B| = k$. Following a construction of G. Elek, we define the *graph of snapshots* $X = (V, E)$ of Z as follows. The set of vertices are given by all the balls $\cup_k \mathcal{B}_k$ and we let a pair of balls $B, B' \in V(E)$ define an edge of X if $B \neq B'$, $||B| - |B'|| \leq 1$ and if $B \cap B' \neq \emptyset$. We endow X with the length metric which makes each edge isometric to $[0, 1]$.

Proposition 6.10 (Bourdon & Pajot [BP4])**.** *The space X of snapshots of Z is hyperbolic in the sense of Gromov, and there exists a visual distance d_v of parameter* 1 *on ∂X and a bi-Lipschitz homeomorphism $(\partial X, d_v) \to (Z, d_Z)$. Moreover, the space X is well-defined up to quasi-isometry.*

This construction is very useful when analyzing the quasiconformal geometry of a compact metric space. In particular, M. Carrasco-Piaggio relied on this space to describe the Ahlfors-regular conformal gauge of a metric space and to establish Theorem 5.19. This approach is reminiscent to P. Pansu's coarse quasiconformal structures [Pan4]. The space of snapshots is also used by J. Lindquist to define other types of combinatorial moduli which remind, in the way they are constructed, P. Pansu's coarse moduli [Lin].

Relationships between real analysis and hyperbolic geometry, based on this space of snapshots, are described by S. Semmes in an appendix of [Gro5].

6.5 Applications to rigidity

We describe some applications which follow from our previous discussions. The first application of quasi-Möbius mappings to hyperbolic geometry is the celebrated Mostow rigidity theorem [Mos1, Mos2]. It has opened a whole body of research. We refer to M. Bourdon's survey for more details and references on the material presented below and for other related works [Bou7].

6.5.1 Compact locally symmetric spaces of rank one

Rank one symmetric spaces are classified into three infinite families and one exceptional space —the Cayley plane built from the division algebra of Cayley numbers. We explain the construction of the three infinite families and refer to [Mos2] for the Cayley plane. Let $\mathbb{K}$ denote either the field of real, complex or quaternionic numbers. Fix $n \geq 2$. We consider on the (right) $\mathbb{K}$-module $\mathbb{K}^{n+1}$ a quadratic form q of signature $(n,1)$ and we let $Y^n_{\mathbb{K}} = q^{-1}(\{-1\})$. The symmetric space $X^n_{\mathbb{K}}$ can then be defined by the set of $\mathbb{K}$-lines intersecting $Y^n_{\mathbb{K}}$; it turns out that q induces a Riemannian metric on $X^n_{\mathbb{K}}$. We will also denote by $X^2_{\mathbb{K}}$ the Cayley plane where $\mathbb{K}$ will represent the division algebra of Cayley numbers. See [Bou7] and the references therein for more details.

Conformal and topological dimensions enable us to distinguish these symmetric spaces.

Theorem 6.11. *The boundary $\partial X^n_{\mathbb{K}}$ has topological dimension $n \dim_{\mathbb{R}} \mathbb{K} - 1$ and is conformally modeled on $G_{\mathbb{K}}$ and its (Ahlfors-regular) conformal dimension satisfies*

$$\mathrm{confdim}_{(AR)} \partial X^n_{\mathbb{K}} = (n+1)\dim_{\mathbb{R}} \mathbb{K} - 2\,.$$

If $X^n_{\mathbb{K}}$ is different from the Poincaré plane, then its boundary is a Loewner space, and if $\mathbb{K} \neq \mathbb{R}$, then it is endowed with the Carnot-Carathéodory structure described in § 5.5.3.

We now state Mostow's rigidity theorem.

Theorem 6.12 (Mostow)**.** *Let M and N be locally symmetric and closed Riemannian manifolds. Assume that $\dim M \geq 3$ and that both manifolds have isomorphic fundamental groups. Then M and N are isometric.*

We focus on rank one symmetric spaces. There are now several ways to approach this theorem, see [Bou7] for more details. One relies on the methods developed in this survey: the isomorphism defines a quasi-isometry of the fundamental groups, hence of the universal covers. This quasi-isometry extends as an equivariant quasi-Möbius mapping at infinity. Theorem 6.11 implies at once that the universal covers have to be isometric. Then the conformal elevator principle can be used to prove that this equivariant quasi-Möbius map is actually genuinely Möbius by using that quasi-Möbius maps are differentiable almost everywhere in dimension at least two. This proves that M and N are isometric (since Möbius transformations define isometries of the rank one symmetric spaces).

In dimension 2, the theorem does not hold since quasi-Möbius maps are not always differentiable almost everywhere. If we start with two hyperbolic closed surfaces $\mathbb{H}^2/G_1$ and $\mathbb{H}^2/G_2$ with isomorphic fundamental groups, then the argument above provides us with an equivariant quasi-Möbius mapping $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that $h \circ G_1 = G_2 \circ h$ by identifying $\mathbb{H}^2$ with the unit disk $\mathbb{D}$ and $\partial\mathbb{H}^2$ with the unit circle $\mathbb{S}^1$. Mostow's theorem then becomes

Theorem 6.13 (Kuusalo [Kuu])**.** *Either h is the restriction of a Möbius transformation so that both surfaces are isometric or h is completely singular.*

See also [Bwn] for a different approach. Further extensions to Fuchsian groups of the first kind appear in [AZ, Bis1, Bis2] and in higher dimension in [Sul1].

Building on Theorem 5.27, P. Pansu obtains the following group-free strengthened rigidity theorem [Pan5].

Theorem 6.14 (Pansu)**.** *Any quasi-isometry of $X^n_{\mathbb{K}}$ lies at bounded distance from an isometry, provided $\mathbb{K}$ denotes the field of quaternionic numbers or the division algebra of Cayley numbers.*

The main point here is that $\partial X^n_{\mathbb{K}}$ is modeled on the Heisenberg group $G_{\mathbb{K}}$ and that there is a canonical correspondence between Möbius transformations of the boundary and isometries of the symmetric spaces.

Combined with Theorem 4.6 and Theorem 5.26, we obtain the following quasi-isometric rigidity of fundamental groups of closed locally symmetric groups of negative curvature.

Corollary 6.15 (quasi-isometric rigidity of symmetric spaces)**.** *Let G be a finitely generated group quasi-isometric to a rank-one symmetric space $X^n_{\mathbb{K}}$. Then there exists a short exact sequence*

$$1 \to F \to G \to H \to 1$$

where F is finite and H is a subgroup of isometries of $X^n_{\mathbb{K}}$ which contains a finite index subgroup Γ such that $X^n_{\mathbb{K}}/\Gamma$ is a closed manifold.

Mostow's rigidity theorem has been the starting point of many generalizations. One of them consists in looking for a characterization of locally symmetric closed manifolds among Riemannian manifolds. This was achieved by U. Hamenstädt as follows

Theorem 6.16 (Hamenstädt [Ham2])**.** *Let M be a closed Riemannian manifold of dimension at least two and of maximum sectional curvature (-1). Let us assume that any geodesic of its universal covering belongs to an isometric copy of the Poincaré plane. Then M is locally symmetric.*

The assumption provides a lot of rectifiable curves on the boundary at infinity of the universal covering $\widetilde{M}$, yielding a Loewner structure on $\partial\widetilde{M}$. By looking at the maximal isometrically embedded copies of real hyperbolic spaces, she is able to recognize the Carnot-Carathéodory structure coming from a symmetric space. An analogous characterization among homogeneous spaces was established by C. Connell [Con].

Another kind of generalization of Mostow rigidity consists in allowing the fundamental group of a closed locally symmetric space (of rank one) to act on a more general space such as CAT(-1) spaces that we briefly define. Let X be a geodesic metric space and T be a geodesic triangle. A *comparison triangle* in the Poincaré plane $\mathbb{H}^2$ is a geodesic triangle $T' \subset \mathbb{H}^2$ together with a map $f_T : T' \to T$ which is isometric on each edge. The space X is a *CAT(-1) space* if, for any geodesic triangle the map f_T is 1-Lipschitz, meaning that the triangles in X are thinner than in hyperbolic space. These spaces are hyperbolic in the sense of Gromov but enjoy much stronger properties [Bou1]. Examples include Hadamard manifolds with sectional curvature bounded above by (-1).

Generalizing G. Mostow's theorem and further work of U. Hamenstädt [Ham1], M. Bourdon establishes the following rigidity of CAT(-1) spaces.

Theorem 6.17 (Bourdon [Bou2]). *Let $M = X^n_{\mathbb{K}}/G$ be a rank one locally symmetric closed manifold of dimension at least three and with fundamental group G. If G acts geometrically on a geodesically complete CAT(-1) space X, then X is isometric to $X^n_{\mathbb{K}}$.*

In the real case, Theorem 5.13 leads to another generalization. In this direction, we may associate to any hyperbolic metric space a coarse notion of curvature following M. Bonk and T. Foertsch [BF]. Let (X, w) be a pointed hyperbolic space. We will say that X has *asymptotic upper curvature* (-1) if there is a constant $c > 0$ such that, for all $x_0, \ldots, x_n$,

$$d(w, [x_0, x_n]) \geq \min_{0 \leq j < n} d(w, [x_j, x_{j+1}]) - \log n - c\,.$$

It can be shown that this condition implies the existence of visual distances with visual parameters for all values $\varepsilon \in (0, 1)$. The case $\varepsilon = 1$ is not necessarily attained.

Theorem 6.18 (Kinneberg [Kin]). *Let G be a group acting geometrically on a hyperbolic proper space X with asymptotic curvature (-1) and with boundary homeomorphic to $\mathbb{S}^{n-1}$, $n \geq 3$. Then*

$$\lim_{R \to \infty} \frac{1}{R} \log \sharp\{g \in G,\ g(w) \in B(w, R)\} \geq n - 1$$

and equality holds if and only if there is a $(1, c)$-quasi-isometry between X and $X^n_{\mathbb{R}}$ conjugating the action of G to that of a group of isometries.

One of the difficulties of the proof is to show, in the case of equality, that there exists a visual distance with visual parameter $\varepsilon = 1$. This is achieved by using a length-volume estimate on high-dimensional cubes due to W. Derrick [Der]. From there, K. Kinneberg shows how to reduce the proof to Theorem 5.13.

There are several features which make rank one symmetric spaces so special and rigid. We list three properties which play an important role.

1. We may endow its boundary with a visual distance which is Q-regular and Q-Loewner (for some $Q > 1$) so that quasi-Möbius maps enjoy many properties.
2. With this structure, Möbius transformations are exactly the boundary values of isometries of the symmetric space.
3. Last but not least, Möbius transformations can be recognized from quasi-Möbius maps using ergodic properties of geodesic flows —an observation due to D. Sullivan [Sul2], see [Bou7] for details.

Fuchsian buildings are other examples of hyperbolic spaces which enjoy these properties as was shown by M. Bourdon, H. Pajot [Bou3, BP1, BP2] and X. Xie [Xie1]. We refer to these papers and [Bou7]. It turns out that the cotangent bundle has dimension 1, so that every quasi-Möbius map is conformal by Theorem 5.31 and actually Möbius. Therefore, any quasi-isometry lies at bounded distance from an isometry as in Theorem 6.14.

6.5.2 Homogeneous spaces of negative curvature

Homogeneous manifolds of negative sectional curvature have been classified by E. Heintze [Het2]. They are all isometric to solvable Lie groups G with a left-invariant Riemannian metric of the form $G = N \rtimes_\alpha \mathbb{R}$; the group N is a simply connected nilpotent Lie group and the action of $\mathbb{R}$ on N is given by a derivation α on the Lie algebra of N, the eigenvalues of which have positive real parts. Let us call such a group G a *Heintze group*. According to Y. Cornulier, we may always assume that the eigenvalues of the derivation are all positive and real, up to quasi-isometry [Cor1]: we then say that G is *purely real*. When (N, α) is a Carnot group, then G is a Heintze group of *Carnot type*; note that this need not be always the case.

The boundary of a Heintze group ∂G is quasi-Möbius equivalent to the Alexandrov compactification of N, cf. Proposition 2.2. The rank 1 symmetric spaces distinguish themselves as the only homogeneous spaces which admit finite volume quotients [Het1]. It turns out that any isometry fixes the special point ∞ in the non-symmetric case.

Following M. Bourdon's exposition [Bou7], there are three main conjectures governing our interests.

Conjecture 6.19. *Let G be a purely real Heintze group which is not quasi-isometric to a symmetric space.*

1. **Pointed sphere conjecture.**— *Any self-quasi-isometry of G preserves the point at infinity.*

2. **Quasi-isometric classification.**— *Any purely real Heintze group quasi-isometric to G is isomorphic to G.*

3. **Quasi-isometric rigidity.**— *Any quasi-isometry between two Heintze groups (quasi-isometric to G) lies at bounded distance from a $(1, C)$-quasi-isometry.*

The conjectures all concern quasi-Möbius maps at infinity. The first says that any quasi-Möbius map of ∂G fixes the special point ∞, the second says that if the boundary of another group is quasi-Möbius equivalent, then they are Möbius equivalent and the third is that any quasi-Möbius map is bi-Lipschitz. For a general survey on the classification of locally compact groups including Heintze groups, see [Cor2].

The pointed sphere conjecture was finally solved by M. Carrasco Piaggio when G is not of Carnot type [CP3]; the particular case of a diagonalizable derivation was previously dealt with by P. Pansu [Pan4] and X. Xie had dealt with the case N Abelian or the real Heisenberg groups (see [Xie2, Xie3, Xie4] and the references therein). The main idea of the proof is to show the existence of a foliation on $N \subset \partial G$ by rectifiable curves (for a visual metric) preserved by quasi-Möbius maps. This foliation can be constructed as follows. Let μ denote the smallest eigenvalue of α and let $H < G$ be the closed connected subgroup whose Lie algebra is generated by the μ-eigenvectors belonging to the μ-Jordan blocks of maximal dimension. Then H is a proper subgroup when (N, α) is not a Carnot group and any quasi-Möbius map preserves the left cosets of H. Nonetheless, the derivation restricted to H defines a Carnot structure. The latter property implies that being a Heintze group of Carnot type is a quasi-isometry invariant [CP3].

Remark 6.20. *With the above notation, if the dimension m of the largest Jordan blocks associated to μ satisfies $m \geq 2$, then M. Carrasco Piaggio shows that the conformal dimension of the boundary ∂G is not attained.*

When N is a Carnot group, then $H = G$ so H cannot be used to find some special curves. Note also that ∂G is also a Loewner space so carries a lot of rectifiable curves in all directions.

The quasi-isometric classification was established by P. Pansu for purely real Carnot-type Heintze groups [Pan5], cf. Theorem 5.25. When N is Abelian, X. Xie proved that the Jordan decomposition of the derivations of two quasi-isometric purely real Heintze group (of Abelian type) are proportional, implying they are isomorphic [Xie2]; this completed the classification initiated by P. Pansu assuming the derivations are diagonalizable [Pan7]. X. Xie also worked out the case of real Heisenberg groups endowed with diagonalizable derivations [Xie3].

X. Xie observed that the quasi-isometric rigidity can be obtained if quasisymmetric maps preserve some foliations of the nilpotent groups: this enabled him to treat the case of Heintze groups of Abelian type and the case of Heisenberg groups with diagonalizable derivations. This idea was also exploited by M. Carrasco Piaggio who proved that any self-quasi-isometry of a purely real Heintze group which is not of Carnot type is bi-Lipschitz at infinity by using the subgroup H mentioned above [CP3].

More generally, E. Le Donne and X. Xie formalized the idea in a general setting that quasisymmetric maps which preserve foliations are bi-Lipschitz [LDX]. They apply this general result to prove that quasisymmetric maps are bi-Lipschitz in the setting of *reducible* Carnot groups. This means that the horizontal space V_1 contains a nontrivial proper linear subspace invariant under the automorphisms which preserve the strata. This subspace generates a Lie subgroup (of Carnot type) which defines the needed foliation.

The quasi-isometric rigidity implies new generalizations of Theorem 4.6.

Theorem 6.21 (Xie [Xie2])**.** *Let G be a purely real Heintze group of Abelian type but not of Carnot type. Assume that Γ is a group of uniform quasi-Möbius maps acting on ∂G. If the diagonal action on the set of distinct triples of ∂G is cocompact, then Γ is conjugate to a so-called group of almost homotheties on N.*

An almost homothety of N is a bi-Lipschitz map which is a perturbation of the identity preserving the Jordan decomposition of the derivation; see [DP] for a precise definition. The idea of the proof is to first remark that Γ fixes the point at ∞ is a group of bi-Lipschitz maps whose action is cocompact on the set of distinct pairs. Work of T. Dymarz and I. Peng then enables him to conclude [DP].

Remark 6.22. *Extensions of these results to model filform groups and millefeuille spaces can be found in [Dym, DX].*

6.5.3 Low dimensional topology and hyperbolic manifolds

A *Kleinian group* is a discrete subgroup of $\mathbb{P}SL_2(\mathbb{C})$ which we view as acting both on hyperbolic 3-space $\mathbb{H}^3$ via orientation-preserving isometries and on the Riemann sphere $\widehat{\mathbb{C}}$ via Möbius transformations. Since Poincaré introduced them for solving differential equations with algebraic coefficients at the end of the nineteenth century [Poi], Kleinian groups have continuously drawn a lot of attention, playing a prominent role in complex analysis, conformal dynamical systems, hyperbolic geometry, Teichmüller theory and low dimensional topology, see for instance [Kln, Ber, Sul2, Thu]. The subclass of *convex-cocompact Kleinian groups* is particularly relevant for the topology of 3-dimensional manifolds. These are finitely generated Kleinian groups G for which there is a convex subset $\mathcal{C} \subset \mathbb{H}^3$ invariant under G such that $\mathcal{C}/G$ is compact. In particular, a *cocompact* Kleinian group is convex-cocompact with $\mathcal{C} = \mathbb{H}^3$. Convex-cocompact Kleinian groups are word hyperbolic and their boundaries coincide with their limit sets. A central conjecture in low dimension is the following problem.

Conjecture 6.23. *A word hyperbolic group with planar boundary contains a finite index subgroup isomorphic to a convex-cocompact Kleinian group.*

This would imply that we may drop the knowledge of the conformal gauge in Paulin's characterization of a word hyperbolic group when its boundary is planar. Theorem 3.12 provides a positive answer when the boundary is homeomorphic to a simple closed curve.

The following two well-known conjectures can be derived from the above problem by specifying the boundary of the group:

Conjecture 6.24. *Let G be a word hyperbolic group.*

- **(Cannon, [Can1, Conjecture 11.34])** *If ∂G is homeomorphic to S^2, then G contains a finite index subgroup isomorphic to a cocompact Kleinian group.*
- **(Kapovich and Kleiner, [KK, Conjecture 6])** *If ∂G is homeomorphic to the Sierpiński carpet, then G contains a finite index subgroup isomorphic to the fundamental group of a compact hyperbolic 3-manifold with non-empty totally geodesic boundary.*

Knowledge on the Ahlfors regular conformal dimension would provide a positive answer to Conjecture 6.23.

Theorem 6.25. *Let G be a non-elementary hyperbolic group with planar boundary.*

1. *If ∂G is homeomorphic to S^2, and if there is an Ahlfors regular distance in its gauge of minimal dimension, then G contains a finite index subgroup isomorphic to a cocompact Kleinian group.*
2. *If ∂G is non-homeomorphic to the sphere and if $\mathrm{confdim}_{AR}(G) < 2$, then G is virtually isomorphic to a convex-cocompact Kleinian group.*

The first part is due to M. Bonk and B. Kleiner [BnK3]; when G is a Coxeter group, M. Bourdon and B. Kleiner were able to check its assumptions [BdK1]. The second statement is proved in [Haï3] where other sufficient conditions can be found. We draw the following corollary.

Corollary 6.26. *Let G be a non-elementary hyperbolic group and let ∂G be endowed with a metric from its conformal gauge. If there exists a quasisymmetric embedding of ∂G into $\widehat{\mathbb{C}}$, then G is virtually isomorphic to a convex-cocompact Kleinian group.*

If ∂G is homeomorphic to $\widehat{\mathbb{C}}$, then this follows from Theorem 4.5. In the other case, the proof boils down to essentially establishing that the quasisymmetric embedding maps the boundary ∂G to a porous subset Λ of the Riemann sphere i..e, any disk D centred on a point of Λ contains a ball of definite radius disjoint from Λ. This implies that the Ahlfors-regular conformal dimension is strictly less than two, so Theorem 6.25 applies.

Corollary 6.26 provides us with a positive answer to a weaker and somewhat intermediate conjecture: *a group quasi-isometric to a convex-cocompact Kleinian group contains a finite-index subgroup isomorphic to a convex-cocompact Kleinian group.* It also enables us to restate Conjecture 6.23 in analytic terms:

Conjecture 6.27. *If the boundary of a word hyperbolic group is planar, then it admits a quasisymmetric embedding in the Riemann sphere $\widehat{\mathbb{C}}$, when equipped with a metric of its conformal gauge.*

We note that these conjectures do not hold in higher dimensions. M. Gromov and W. Thurston have constructed closed manifolds in any dimension $d \geq 4$ of arbitrarily pinched negative curvature the fundamental groups of which are not quasi-isometric to any rank one symmetric space [GT, Pan1]. There are also counter-examples due to Y. Benoist, which are constructed as word hyperbolic groups acting projectively, properly discontinuously and cocompactly on convex domains [Ben].

7 Actions on functional spaces

The basic question of this section is the following: given a compact metric space X, determine which functional spaces $\mathcal{F} = \{f : X \to \mathbb{R}\}$ are invariant under precomposition by quasi-Möbius mappings? We will also see that some of these functional spaces characterize the conformal gauge of compact metric spaces, cf. Theorem 7.11. A detailed review of these questions will appear in the survey [KSS]. We end up this section with a couple of words on ℓ^p-cohomology which is the counterpart of these functional spaces from the hyperbolic point of view.

7.1 Poincaré inequality spaces

The setting of this section will be compact metric measure spaces (X, d_X, μ) which carry many rectifiable curves. The latter is best expressed in terms of Poincaré inequalities in the sense of Heinonen and Koskela [HnK2].

Standard Poincaré inequalities provide control on the oscillations of a smooth function in terms of the mean of its gradient. Let $u : \Omega \to \mathbb{R}$ be a smooth function defined in a domain of some Euclidean space, and let $B \subset \Omega$ be a ball. Then, for any $p \geq 1$,

$$\int_B |u - u_B| dx \leq C \mathrm{diam} B \left(\fint_B \|\nabla u\|^p dx \right)^{1/p}$$

where

$$u_B = \fint_B u dx = \frac{1}{|B|} \int_B u dx$$

denotes the mean value of $u|_B$.

To state such an inequality on a metric space, we first need to find a substitute for the gradient of a function. This is done on the base of the mean value theorem. Let $u : X \to \mathbb{R}$ be a function. An *upper gradient* for u is a measurable function $g : X \to [0, \infty]$ such that, for $x, y \in X$ and any rectifiable curve γ joining x to y,

$$|u(x) - u(y)| \leq \int_\gamma g(z) ds(z)$$

where the integral over curves has been defined in § 2.3. Note that any function admits $g \equiv \infty$ as an upper gradient.

We say that the metric measure space (X, d_X, μ) *supports a* $(1,p)$*-Poincaré inequality* if there exist constant $C \geq 0$ and $\tau \geq 1$ so that

$$\int_B |u - u_B| d\mu \leq C \mathrm{diam} B \left(\fint_{\tau \cdot B} g^p d\mu \right)^{1/p}$$

holds for any continuous function $u : X \to \mathbb{R}$ and any upper gradient g of u. On the right hand side, the mean is taken over the slightly larger ball $\tau \cdot B$, which takes into account the fact that the space X need not be geodesic. Finally, we note that the Hölder inequality implies that a space supports a $(1,q)$-Poincaré inequality for $q > p$ as soon as it supports a $(1,p)$-Poincaré inequality.

Definition 7.1. *A PI space is a metric measure space (X, d, μ) such that μ is doubling i.e., $\mu(2B) \leq C\mu(B)$ for all balls $B \subset X$ and uniform constant $C \geq 1$ and which carries a $(1,p)$-Poincaré inequality for some $p \geq 1$. We say that X is a p-PI space to emphasize the power p in the Poincaré inequality.*

Most of the analysis in Euclidean spaces can be carried out for PI spaces. In particular, we have the following properties.

Theorem 7.2. *Let X be a p-PI space. The following properties hold.*

1. *There is a constant $C_\ell \geq 1$ such that any pair of points $x, y \in X$ can be joined by a curve γ such that $\ell(\gamma) \leq C_\ell d(x,y)$.*

2. *If X is Q-Ahlfors regular and $p \leq Q$, then X is a Loewner space.*

3. *The space X is a differentiability space.*

This result follows from works of J. Heinonen and P. Koskela [HnK2] and J. Cheeger [Che]. We may also refer to [HKST2], Chapters 8 and 13 for a more detailed treatment.

7.2 Sobolev spaces

Let (X, d, μ) be a PI space. If $u : X \to \mathbb{R}$ is a measurable function, we say that g is a *p-weak upper gradient* if

$$|u(x) - u(y)| \leq \int_\gamma g(x) ds(x)$$

holds for p-almost every curve i.e, the family of curves for which this does not hold has vanishing p-modulus.

The *Newtonian space* $W^{1,p}(X)$ is the set of measurable functions $u : X \to \mathbb{R}$ which admits a p-weak upper gradient g in L^p. This notion was introduced by N. Shanmugalingam [Sha]. When $X = \mathbb{R}^n$, then $W^{1,p}(X)$ coincides with the classical Sobolev space $W^{1,p}(\mathbb{R}^n)$ of measurable functions which admit weak derivatives in L^p. This leads to a Banach space with the norm

$$\|u\|_{1,p} = \|u\|_p + \inf_g \|g\|_p$$

where g ranges over all weak upper gradients of u.

In PI spaces, Newtonian spaces have a very rich structure and have many alternative characterizations. In particular, they concide with the Hajłasz-Sobolev spaces previously defined by P. Hajłasz [Haj]. For any $s \in (0,\infty)$, and measurable function u on X, a function $g : X \to \mathbb{R}_+$ is called an *s-gradient* of u if there exists a set E of measure 0 such that, for all $x, y \in X \setminus E$,

$$|u(x) - u(y)| \le [d(x,y)]^s [g(x) + g(y)] .$$

In the Euclidean setting, g corresponds to the maximal function of $|\nabla u|$. Let $\dot{M}^{1,p}(X)$ denote the homogeneous space of measurable functions u such that

$$\|u\|_{\dot{M}^{1,p}(X)} = \inf_g \|g\|_{L^p(X)} \text{ is finite}$$

where the infimum is taken over all p-weak upper gradients g of u. We also define $M^{1,p}(X)$ as the set of measurable functions u such that

$$\|u\|_{\dot{M}^{1,p}(X)} = \|u\|_{L^p(X)} + \|u\|_{\dot{M}^{1,p}(X)} \text{ is finite.}$$

The advantage of the latter definition is that it makes sense, even in metric spaces with no rectifiable curves.

Theorem 7.3 (Shanmugalingam). *If X is a q-PI space and $p > q$, then $M^{1,p}(X)$ and $W^{1,p}(X)$ are isomorphic as Banach spaces.*

Regarding the invariance of Sobolev maps, we have the following result which generalizes the Euclidean setting.

Theorem 7.4 (Koskela & McManus [KoM]). *Let (X, d_X, μ) be a compact Q-regular Q-PI space for some $Q > 1$ and (Y, d_Y, ν) be linearly locally connected and Q-regular. If $f : X \to Y$ is quasi-Möbius, then $u \in W^{1,Q}(Y)$ if and only if $u \circ f \in W^{1,Q}(X)$.*

Notes.— The book [HKST2] contains ample background and information on Sobolev mappings in metric spaces.

7.3 Besov and Triebel-Lizorkin spaces

Besov and Triebel-Lizorkin spaces have been introduced as interpolation spaces between the Lebesgue spaces L^p, the space BMO, the Sobolev spaces, the Hardy-Sobolev spaces, etc. As such they form an important tool for harmonic analysis. The literature from the last fifty years or so contains a large amount of different but useful characterizations of Triebel-Lizorkin and Besov spaces; we just refer to the book H. Triebel [Tri] for simplicity.

We describe here their definition following the work of P. Koskela, D. Yang and Y. Zhou which provides us with an equivalent definition which makes sense in general metric spaces [KYZ]. We then study their invariance under quasi-Möbius mappings. All the results in this section are due to P. Koskela, D. Yang and Y. Zhou, unless explicitly stated. The results are not always stated in the most general form; we refer to [KYZ] for that, and to the references therein.

We first adapt the notion of s-gradients as follows. A sequence of nonnegative functions $\vec{g} = (g_k)_{k\in\mathbb{Z}}$ is called a *fractional s-Hajłasz gradient of u* if there exists a set E of measure 0 such that, for all $x, y \in X\setminus E$ such that $2^{-(k+1)} \le d(x,y) \le 2^{-k}$,

$$|u(x) - u(y)| \le [d(x,y)]^s [g(x) + g(y)]\,.$$

We first define

$$\|\vec{g}\|_{\ell^q} = \left(\sum_{k\in\mathbb{Z}} |g_k|^q\right)^{1/q}$$

for $1 \le q < \infty$ and $\|\vec{g}\|_{\ell^\infty} = \sup_k |g_k|$; moreover we define

$$\|\vec{g}\|_{L^p(X,\ell^q)} = \|\ \|\vec{g}\|_{\ell^q}\ \|_{L^p(X)} \quad \text{and} \quad \|\vec{g}\|_{\ell^q(L^p(X))} = \|\ \|g_k\|_{L^p(X)}\ \|_{\ell^q}\,.$$

Definition 7.5 (homogeneous Hajłasz-Triebel-Lizorkin spaces)**.** *Let (X, d, μ) be a metric measure space, $s \in (0,\infty)$ and $q \in (0,\infty]$.*

- *If $p \in (0,\infty)$, the homogeneous Hajłasz-Triebel-Lizorkin space $\dot{M}^s_{p,q}(X)$ is the space of all measurable functions u such that $\|u\|_{\dot{M}^s_{p,q}(X)}$ is finite, where*

  $$\|u\|_{\dot{M}^s_{p,q}(X)} = \inf_{\vec{g}} \|\vec{g}\|_{L^p(X,\ell^q)}$$

  and where the infimum is taken over all fractional s-Hajłasz gradients of u.

- *If $p = \infty$, the homogeneous Hajłasz-Triebel-Lizorkin space $\dot{M}^s_{\infty,q}(X)$ is the space of all measurable functions usuch that $\|u\|_{\dot{M}^s_{\infty,q}(X)}$ is finite, where*

  $$\|u\|_{\dot{M}^s_{\infty,q}(X)} = \inf_{\vec{g}} \sup_{k\in\mathbb{Z}} \sup_{x\in X} \left\{\sum_{j\ge k} \fint_{B(x,2^{-k})} [g_j(y)]^q d\mu(y)\right\}^{1/q}$$

  when $q < \infty$; when $q = \infty$, $\|u\|_{\dot{M}^s_{\infty,\infty}(X)} = \|\vec{g}\|_{L^\infty(X,\ell^\infty)}$. In both cases, the infimum is taken over all fractional s-Hajłasz gradients of u.

Definition 7.6 (homogeneous Hajłasz-Besov spaces)**.** *Let (X, d, μ) be a metric measure space, $s \in (0,\infty)$ and $p, q \in (0,\infty]$. The homogeneous Hajłasz-Besov space $\dot{N}^s_{p,q}(X)$ is the space of all measurable functions u such that $\|u\|_{\dot{N}^s_{p,q}(X)}$ is finite, where*

$$\|u\|_{\dot{N}^s_{p,q}(X)} = \inf_{\vec{g}} \|\vec{g}\|_{\ell^q(L^p(X))}$$

We introduce another version of Besov spaces, where the norm might be simpler to grasp and which was used by M. Bourdon and H. Pajot [BP4, Bou5].

Definition 7.7 (homogeneous Besov spaces)**.** *Let (X,d,μ) be a Q-Ahlfors regular metric space, $s\in(0,\infty)$ and $p,q\in[1,\infty)$. The homogeneous Besov space $\dot{B}^s_{p,q}(X)$ is the space of all measurable functions $u\in L^p_{loc}(X)$ such that $\|u\|_{\dot{B}^s_{p,q}(X)}$ is finite, where*

$$\|u\|_{\dot{B}^s_{p,q}(X)}=\left\{\int_0^\infty\left(\int_X \fint_{B(x,t)}|u(x)-u(y)|^p d\mu(y)d\mu(x)\right)^{q/p}\frac{dt}{t^{1+sq}}\right\}^{1/q}.$$

The homogeneous Besov space $\dot{B}^s_p(X)$ is the space of all measurable functions $u\in L^p_{loc}(X)$ such that $\|u\|_{\dot{B}^s_p(X)}$ is finite, where

$$\|u\|_{\dot{B}^s_p(X)}=\left\{\int_{X\times X}\frac{|u(x)-u(y)|^p}{[d(x,y)]^{sp+Q}}d\mu(x)d\mu(y)\right\}^{1/p}.$$

Fubini's theorem implies that $B^s_p(X)=B^s_{p,p}(X)$ (with equivalent norms). The parameter s measures the *smoothness* of the function. For $s\in(0,1)$, and replacing L^p-norms by L^∞, we may observe that $B^s_\infty(X)$ coincides with the space of s-Hölder functions.

Proposition 7.8. *Let (X,d,μ) be an Ahlfors regular metric space. We have the following identifications.*

1. *If $s\in(0,\infty)$ and $p\in(0,\infty]$ then $\dot{M}^s_{p,\infty}(X)=\dot{M}^{s,p}(X)$.*
2. *If $s\in(0,\infty)$ and $p,q\in(0,\infty]$ then $\dot{N}^s_{p,q}(X)=\dot{B}^s_{p,q}(X)$.*
3. *If $s\in(0,\infty)$ and $p\in[1,\infty)$ then $\dot{M}^s_{p,p}(X)=\dot{B}^s_p(X)$.*

The identification between the different forms of Besov spaces is established in [GKZ].

Before stating the main results of this section, let us observe that if u belongs to one of these functional spaces and f is quasi-Möbius, then it is not clear at all —and sometimes wrong— that $u\circ f$ is even measurable. So we will say that f *induces a bounded operator* between functional spaces $\mathcal{F}_1$ and $\mathcal{F}_2$ if $u\in\mathcal{F}_2$ has a measurable representative v such that $v\circ f$ is in $\mathcal{F}_1$ and $\|v\circ f\|_1\le C\|u\|_2$.

Also, we note that in a Q-regular space, the invariance of these spaces under scaling is only satisfied when $p=Q/s$.

Theorem 7.9 (Quasi-Möbius invariance)**.** *Let X, Y be Q_X- and Q_Y-regular compact metric spaces respectively and let $f:X\to Y$ be a quasi-Möbius homeomorphism.*

1. *Let $s_X\in(0,Q_X)$ and $s_Y\in(0,Q_Y)$ satisfy $Q_X/s_X=Q_Y/s_Y$. Then f induces a bounded operator between $\dot{B}^{s_X}_{Q_X/s_X}(X)$ and $\dot{B}^{s_Y}_{Q_Y/s_Y}(Y)$.*
2. *If $Q_X=Q_Y=Q$ and X is Q-Loewner for some $Q>1$, then, for any $s\in(0,1]$ and for all $q\in(0,\infty]$, f induces a bounded operator between $\dot{M}^s_{Q/s,q}(X)$ and $\dot{M}^s_{Q/s,q}(Y)$.*

The Besov case was established by M. Bourdon and H. Pajot in [BP4]. When $s > 1$, the Hajłasz-Triebel-Lizorkin spaces $\dot{M}^s_{Q/s,q}(X)$ turn out to be trivial when X is Q-regular and Q-Loewner [GKZ]. These invariance properties come with converse statements:

Theorem 7.10 (Regularity of composition operators)**.** *Let X, Y be Q-regular connected compact metric spaces for some $Q > 1$, with Y linearly locally connected, and let $f : X \to Y$ be a homeomorphism.*

1. *Assume that $0 < s < 1$, $1 < q < \infty$. If f defines a bounded operator between $\dot{B}^s_{Q/s,q}(Y)$ and $\dot{B}^s_{Q/s,q}(X)$ and if $q = Q/s$ then f is quasi-Möbius; if $q \neq Q/s$ and X is Q-Loewner, then f is bi-Lipschitz.*

2. *Assume that $0 < s < 1$ and $Q/(Q + s) < q \leq \infty$. If f induces a bounded operator between $\dot{M}^s_{Q/s,q}(X)$ and $\dot{M}^s_{Q/s,q}(Y)$ then f is quasi-Möbius.*

The regularity in the Besov setting was established by M. Bourdon [Bou5] ($q = Q/s$) and by H. Koch, P. Koskela, E. Saksman and T. Soto [KKSS] ($q \neq Q/s$), generalizing [Fer2]. In the Hajłasz-Triebel-Lizorkin setting, this is due to M. Bonk, E. Saksman and T. Soto [BSS]; for $1 < q < \infty$, it was also established in [KKSS]. See also [HcK] for sharper results on Euclidean spaces.

All the proofs proceed similarly for $q = Q/s$. A Besov capacity can be defined with good geometrical bounds (as for the conformal capacity in Q-regular Q-Loewner spaces) by using continuous Besov functions. The boundedness of the operator then implies that the homeomorphism quasi-preserves these capacities, implying it is a quasi-Möbius mapping. The Triebel-Lizorkin spaces can be embedded in some Besov spaces and vice-versa: this enables the authors to reduce this more general case to the previous ones.

7.4 Royden-type algebras

In this section, we show that the functional spaces characterize metric spaces up to quasi-Möbius mappings. If $\mathcal{F}$ is a space of functions as above, then we denote by $A_{\mathcal{F}}$ the set of continuous and bounded functions in the homogeneous space $\dot{\mathcal{F}}$ that we endow with the norm

$$\|f\|_{A_{\mathcal{F}}} = \|f\|_\infty + \|f\|_{\dot{\mathcal{F}}} .$$

This space is reminiscent to Royden's algebras defined for Riemann surfaces [Roy].

Theorem 7.11. *Let X, Y be Q-regular compact metric spaces, for some $Q > 1$, with Y linearly locally connected. Let $\mathcal{F}$ denote one of the following functional spaces: $W^{1,Q}$, $\dot{B}^s_{Q/s,q}$ or $\dot{M}^s_{Q/s,q}$, where $0 < s < 1$ and $Q/(Q + s) < q < \infty$. If there exists an algebra isomorphism $T : A_{\mathcal{F}}(Y) \to A_{\mathcal{F}}(X)$, continuous with respect to the homogeneous norms $\|\cdot\|_{\dot{\mathcal{F}}}$, then there exists a quasi-Möbius homeomorphism $f : X \to Y$ such that T is given by the composition by f. In particular, both X and Y are quasi-Möbius equivalent.*

This result was first established for Sobolev functions by M. Nakai for Riemann surfaces [Nak], L. Lewis for Euclidean domains [Lew] and J.Ferrand in the context of Riemannian manifolds [Fer2]. M. Bourdon took care of the Besov spaces $\dot{B}^s_{Q/s}$ in the context of Ahlfors regular spaces [Bou5].

Remark 7.12. *In the Riemannian settings, the boundedness is expressed in the norms $A_{\mathcal{F}}$. It is not clear that we may consider the same norms in metric spaces. What needs to be proved is that a sequence of maps $(u_n)_n$ in $A_{\mathcal{F}}$, whose homogeneous norms tend to zero, tends to a constant up to a subsequence. The main difference with the Riemannian setting is that, in one case, the norms are defined through gradients which are defined infinitesimally whereas in the other case, upper gradients are used, which are globally defined.*

In view of Theorem 7.10, one has to prove that the isomorphism comes from a composition operator by a homeomorphism. With our assumptions, $(A_{\mathcal{F}}, \|\cdot\|_{A_{\mathcal{F}}})$ is a quasi-Banach algebra with unit which contains the algebra of Lipschitz functions. Therefore, we may separate pairs of points of X by elements from $A = A_{\mathcal{F}}$. One can also prove that, for all $f \in A$,

$$\lim \|f^n\|_{\dot{\mathcal{F}}}^{1/n} = \|f\|_\infty \,.$$

This implies that the operator is also bounded with respect to the $A_{\mathcal{F}}$-norm. All these properties can be checked by looking at the (fractional) Hajłasz gradients.

Assuming X and Y are compact simplifies several points of the proof. For $Z \in \{X, Y\}$, let us consider the set of characters Z^* of A i.e., the collection of continuous and real-valued algebra morphisms $\chi : A \to \mathbb{R}$. It follows that every $\chi \in Z^*$ has norm one and that Z^* is closed in the dual space of A endowed with the weak-* topology, hence is compact (and metrizable). Given an isomorphism $T : A(Y) \to A(X)$, we consider the adjoint $T^* : X^* \to Y^*$ defined by $T^*(\chi) = \chi \circ T$: this is a homeomorphism. The main idea is to relate these compact spaces to X and Y.

We now fix $Z \in \{X, Y\}$. Since $A = A(Z)$ separates points, Z embeds continuously into Z^* via the map $\phi : Z \to Z^*$ defined by $\phi(z)(f) = f(z)$. We wish to prove that $\phi(Z) = Z^*$. To see this, we embed A into the Banach space $\mathcal{C}(Z^*)$ of continuous functions via $\psi : A \to \mathcal{C}(Z^*)$ defined by $\psi(f)(\chi) = \chi(f)$. We note that ψ is an injective algebra morphism. So its image is dense in $\mathcal{C}(Z^*)$ by the Stone-Weierstrass theorem. If $\phi(Z) \neq Z^*$, then we can find a nontrivial continuous function $g \in \mathcal{C}(Z^*)$ such that $g|_{\phi(Z)} \equiv 0$ because Z is compact and Z^* is metrizable. But g is the uniform limit of a sequence $(\psi(f_k))_k$. Since g is vanishing on $\phi(Z)$, we deduce that $(f_n)_n$ tends uniformly to 0 on Z. But, for all k and all $\chi \in Z^*$,

$$|\psi(f_k)(\chi)| = |\chi(f_k)| = \lim_{n\to\infty} |\chi(f_k^n)|^{1/n} \leq \lim_{n\to\infty} \|f_k^n\|_A^{1/n} = \|f_k\|_\infty \,.$$

From this, we deduce that $g \equiv 0$ —a contradiction.

Thus, the adjoint map $T^* : X^* \to Y^*$ provides us with a map $\varphi : X \to Y$ by setting $\varphi(x) = y$ if $T^*\phi(x) = \phi(y)$. Hence, for all $x \in X$,

$$T(f)(x) = \phi(x)(T(f)) = T^*(\phi(x))(f) = \phi(\varphi(x))(f) = (f \circ \varphi)(x) \,.$$

7.5 ℓ^p-cohomology

The systematic study of L^p-cohomology in negatively curved spaces, with an emphasis on negatively curved homogeneous manifolds, was initiated by P. Pansu in a series of papers, e.g. [Pan6, Pan3, Pan7]. One initial motivation was to find global obstructions for the existence of Sobolev inequalities in Riemannian manifolds and for the existence of conformal invariant distances. Other important applications concern the optimization of pinching the sectional curvatures of Riemannian manifolds of negative curvature.

P. Pansu shows that these cohomology groups are quasi-isometry invariants under reasonable conditions and that ℓ^p-classes in degree 1 extend on the boundary at infinity as Besov functions [Pan3]. This is the starting point of a whole field of research connecting large scale quasi-isometry invariants of hyperbolic metric spaces and quasiconformal geometry at infinity, starting with the works of M. Bourdon and H. Pajot [BP4].

We focus on the cohomology groups associated to Ahlfors regular compact metric spaces and refer to [Pan8] for an overview in the context of Riemannian geometry.

7.5.1 Definition on graphs and first properties

Let Z be an Ahlfors regular compact metric space and let us consider the hyperbolic graph of its snapshots X defined in § 6.4. If $f : X^{(0)} \to \mathbb{R}$, let us define $df : X^{(1)} \to \mathbb{R}$ by $df(v,w) = f(w) - f(v)$. Given $p \geq 1$, the first ℓ^p-cohomology group is (isomorphic to)

$$\ell^p H^1(X) = \{f \in \ell^p(X^{(0)}),\ df \in \ell^p(X^{(1)})\}/\ell^p(X^{(0)}) + \mathbb{R}\,.$$

According to [BP4], these groups only depend on the conformal gauge of X. Moreover, a theorem of Strichartz shows that there is a monomorphism $\varphi : \ell^p H^1(X) \hookrightarrow L^p(Z)/\mathbb{R}$ given by taking radial limits of representatives. More precisely,

Theorem 7.13 (Bourdon & Pajot)**.** *The monomorphism $\varphi : \ell^p H^1(X) \hookrightarrow L^p(Z)/\mathbb{R}$ defines an isomorphism between $\ell^p H^1(X)$ and the homogeneous Besov space $\dot{B}_p^{Q/p}(Z)$.*

Let us recall that cohomology spaces are invariant under quasi-isometries and that we have seen above that these Besov spaces are invariant under quasi-Möbius maps. More importantly, it is relatively hard to construct nontrivial Besov functions, whereas cohomology classes turn out to be much more manageable [Bou4, BdK2].

Let us denote by A_p the Royden algebra of the Besov space introduced in the previous paragraph: these are the continuous functions with finite homogeneous $B_p^{Q/p}$-norm. This space was introduced in [Bou4]; see also [Ele, Gro4]. Define the equivalence relation $\sim_p$ on Z by setting $x \sim_p y$ if, for any $u \in A_p$, $u(x) = u(y)$, so that A_p cannot separate those points.

Let us recall that Lipschitz functions are contained in A_p if p is large enough, and let us define two critical exponents.

- Let $p_{\neq 0} = \inf\{p \geq 1,\ A_p \neq \mathbb{R}\}$ so that A_p is non trivial for all $p > p_{\neq 0}$.
- Let $p_{sep} = \inf\{p \geq 1,\ Z/\sim_p = Z\}$ so that we may separate any pair of distinct points by elements of A_p for $p > p_{sep}$.

We have the following properties established by cohomology methods.

Theorem 7.14 (Bourdon & Kleiner [BdK2])**.** *Let Z be a connected compact metric space which contains a large group of uniform quasi-Möbius maps with linear distortion function.*

1. *Equivalence classes of $\sim_p$ are continua.*
2. *We have $p_{sep} = \mathrm{confdim}_{AR} Z$ and $p_{sep} \geq p_{\neq 0}$.*
3. *If Z satisfies the combinatorial Loewner property then $p_{sep} = p_{\neq 0}$.*

When Z is a Loewner space, the equality $p_{\neq 0} = \mathrm{confdim}_{AR} Z$ was established in [BP4]. The control on the cohomology classes enables M. Bourdon and B. Kleiner to construct many examples of word hyperbolic groups which satisfy (or not) the combinatorial Loewner property, and with particularly good control on their Ahlfors-conformal dimension. These examples generalize those from [Bou4], where M. Bourdon constructs word hyperbolic groups with proper and non-trivial $\sim_p$-classes which are isomorphic to limit sets of malnormal quasiconvex subgroups.

Remark 7.15. *In [Bou6], M. Bourdon studies ℓ^p-cohomology of higher degree and also relates their triviality with the Ahlfors-regular conformal dimensions. This enables him to find criteria (and examples) of word hyperbolic groups for which $1 < \dim_{top} \partial G < \mathrm{confdim}_{AR} \partial G$, extending Theorem 5.12 to higher topological dimension.*

7.5.2 Generalizations

Let us note that the work of M. Carrasco Piaggio on the quasi-isometries of negatively curved homogeneous manifolds mentioned in § 6.5.2 relies on the computation of cohomology groups where the decay at infinity satisfies some Orlicz-regularity. The trace at infinity of these classes belong to a so-called Orlicz-Besov space. He then considers the associated Royden algebra and is able to obtain some structure at infinity necessary to exhibit rigidity phenomena [CP3].

Since Besov functions on compact metric spaces Z correspond to ℓ^p-cohomology classes of their space of snapshots X, it is a natural question to understand what are the counterparts of Sobolev functions and more generally functions in the Triebel-Lizorkin spaces of Z in X. These questions were worked out by M. Bonk, E. Saksman and T. Soto [BSk, BSS]. This point of view has enabled them to increase the range on the parameters for the quasi-Möbius invariance.

8 Actions on Sierpiński carpets

The standard Sierpiński carpet S is a self-similar fractal in the plane defined as follows. Let $Q_0 = [0,1] \times [0,1]$ denote the closed unit square. We subdivide Q_0 into 3×3 subsquares of equal size in the obvious way and remove the interior of the middle square. The resulting set Q_1 consists of eight squares of sidelength $1/3$. Inductively, Q_{n+1}, $n \geq 1$, is obtained from Q_n by subdividing each of the remaining squares in the subdivision of Q_n into 3×3 subsquares and removing the interiors of the middle squares. The standard Sierpiński carpet S is the intersection of all the sets Q_n, $n \geq 0$.

It enjoys remarkable topological properties: any planar compact subset of topological dimension 1 admits a topological embedding into S. Moreover, any planar compact connected space of topological dimension 1 that is locally connected and has no local cut-points is homeomorphic to the Sierpiński carpet. We will denote in the sequel a homeomorphic compact set a *carpet*. Here is another characterization. Suppose X is a continuum embedded in the plane. Suppose its complement in the plane has countably many connected components $C_1, C_2, C_3, \ldots$ and suppose:

1. the diameter of C_i goes to zero as $i \to \infty$;
2. the boundary of C_i and the boundary of C_j are disjoint if $i \neq j$;
3. the boundary of C_i is a simple closed curve for each i;
4. the union of the boundaries of the sets C_i is dense in X.

Then X is homeomorphic to the Sierpiński carpet. The group of homeomorphisms of S is very large: it is a Polish topological group which is totally disconnected and one-dimensional. Its action has exactly two orbits: one of them is the union of all simple closed curves which are the boundaries of complementary domains [Bre, Kra].

This compact set appears naturally as Julia sets of rational maps, as limit sets of Kleinian groups, and as boundaries of word hyperbolic groups. In these settings, many natural questions arise from a geometric point of view. In particular, how large is the group of quasi-Möbius maps? They are all homeomorphic. Are they also quasi-Möbius equivalent?

8.1 Rigid carpets

A *round carpet* is a compact subset of the Euclidean sphere whose complementary components are round disks and which is hormeomorphic to the Sierpiński carpet.

We have the following remarkable result:

Theorem 8.1 (Bonk, Kleiner & Merenkov)**.** *Let $X \subset \mathbb{S}^2$ be a round carpet.*

1. *Any quasi-Möbius map $f : X \to X$ is the restriction of a global quasi-Möbius map.*
2. *If X has zero Lebesgue measure, then any quasi-Möbius map is the restriction of a Möbius transformation.*

This theorem applies to any compact subset of the plane quasi-Möbius equivalent to a round carpet: in that broader setting, one obtains a uniformly quasi-Möbius extension. M. Bonk has established the following criterion:

Theorem 8.2 (Bonk)**.** *Let $X \subset \mathbb{S}^2$ be a carpet. It is quasi-Möbius equivalent to a round carpet if the following properties hold*

1. *there exists $K \geq 1$ such that any boundary component is a K-quasicircle;*
2. *there exists $s > 0$ such that $\Delta(C_1, C_2) \geq s$ for any pairs of complementary components.*

The round carpet is unique up to Möbius transformation if the Lebesgue measure of X is zero.

Corollary 8.3. *Let X be a carpet quasi-Möbius equivalent to a round carpet. The group of quasi-Möbius transformations is uniformly quasi-Möbius and discrete.*

In particular, the latter theorem applies to the Sierpiński carpet. But for the latter, we have the following rigidity statement:

Theorem 8.4 (Bonk and Merenkov)**.** *The group of quasi-Möbius maps of S is finite and corresponds to its group of isometries.*

Let us note that the Sierpiński carpet satisfies also the combinatorial Loewner property [BdK1]. In [BM], and in [Zen], the authors prove rigidity for other families of selfsimilar carpets defined by removing subsquares and they show that they are not quasi-Möbius equivalent to one another.

8.2 The geometry of limit sets and Julia sets

Let M be an irreducible, orientable, compact 3-manifold with infinite fundamental group, non-empty and incompressible boundary and no essential properly embedded annuli. There exists a convex-cocompact Kleinian group K such that M is homeomorphic to $\mathbb{H}^3 \cup \Omega_K/K$ and its limit set Λ_K is a round carpet of measure 0.

Proposition 8.5. *Let G be a carpet group quasi-isometric to a Kleinian group and which acts faithfully on its boundary. Then G has finite index in the group G_M of quasi-Möbius homeomorphisms of ∂G. The following properties also hold.*

1. *The group G_M is the unique maximal word hyperbolic group in the quasi-isometry class of G which acts faithfully on its boundary. It is isomorphic to a Kleinian group.*
2. *Whenever G acts geometrically on a proper geodesic metric space, any self-quasi-isometry of X lies at bounded distance from an element of G_M.*
3. *For any group H quasi-isometric to G, H/F is isomorphic to a finite index subgroup of G_M where F is the kernel of the action of H on ∂H.*
4. *For any convex-cocompact Kleinian group K quasi-isometric to G, there is a finite locally isometric covering $p : M_K \to \mathbb{H}^3/G_M$.*

5. Any group quasi-isometric to G is commensurable to G.

M. Bourdon and B. Kleiner provide many examples of Kleinian carpet groups whose limit sets have different geometric features using ℓ^p-cohomology methods [BdK2]. Note that Theorem 5.12 implies that $\text{confdim} G > 1$ for any carpet group.

Theorem 8.6 (Bourdon & Kleiner)**.** *There exists a sequence of carpet groups $(G_n)_n$ such that*

$$\lim_{n\to\infty} \text{confdim}_{AR} G_n = 1\,.$$

Among those groups, some satisfy the combinatorial Loewner property, and others do not.

Let $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a rational map of degree $d\ge 2$. which induces a dynamical system by iteration. The Julia set J_f of f denotes the non-empty compact set of points z such that the family of iterates $(f^n)_n$ is not normal restricted to any neighborhood of z. Denote by $C_f=\{f'=0\}$ the set of critical points and by $P_f = \cup_{c\in C_f}\cup_{n\ge 1} f^n(c)$ its postcritical set. There are many examples of rational maps which have a Julia set with the topology of a carpet. In many instances, a small perturbation of the rational map does not change the topology nor the geometry of the Julia set. It is therefore natural to focus on the so-called postcritically finite rational maps i.e., those for which P_f is finite. For those, we also have rigidity:

Theorem 8.7 (Bonk, Lyubich & Merenkov [BLM])**.** *Let J_f be a carpet Julia set of a postcritically finite rational map. The group of quasi-Möbius transformations is finite, and corresponds to restrictions of Möbius transformations.*

From this, M. Bonk, M. Lyubich and S. Merenkov deduce this surprising consequence.

Corollary 8.8. *The Julia set of a rational map is not quasi-Möbius equivalent to the limit set of a Kleinian group.*

Let us note that carpet limit sets of convex-compact Kleinian groups satisfy the assumptions of Theorem 8.2, and carpet Julia sets of postcritially finite rational maps as well, and, more generally, they share many geometric properties. Thus, this corollary infers that there should be much finer geometric invariants which should help classify conformal gauges.

Remark 8.9. *Theorem 8.7 extends to those so-called semi-hyperbolic rational maps for which no critical point accumulates the boundary of a periodic Fatou component without landing on it [Zen].*

8.3 A flexible carpet

We recall the main example in [Mer] which provides us with an example of a metric carpet with an uncountable group of quasi-Möbius maps.

We first let $Q=[0,1]\times[0,1]$ and consider a slit $Y_0 = Q\setminus(\{1/2\}\times[1/4,3/4])$. The set Y_0 contains 2×2 subsquares of sidelength $1/2$. Remove from Y_0 the corresponding vertical slits to each subsquare and denote by Y_1 the resulting open set. Proceed inductively to define $(Y_n)_n$, where Y_{n+1} is obtained from Y_n by removing middle vertical slits of the 2^{n+1} subsquares in Y_n of sidelength $1/2^{n+1}$.

For each n, endow Y_n with the length metric provided by the Euclidean metric in the plane and let (X_n, d_n) denote the completion of Y_n. Note that each slit becomes a Jordan curve which is a rescaled copy of $\mathbb{R}/\mathbb{Z}$.

There exist natural projections $p_{n+1} : X_{n+1} \to X_n$ for each $n \geq 0$. Let $X = \varprojlim (X_n, p_n)$ denote the projective limit. Note that for any points $x = (x_n)_n$ and $y = (y_n)_n$, the sequence $(d_n(x_n, y_n))_n$ is increasing and bounded, hence convergent. The limit defines a metric d_X on X. Take two copies of X and glue them isometrically along the boundary of the outer square. Denote the result by $\widehat{X}$. This is a sort of annulus. It comes with a projection $p : \widehat{X} \to X_0$.

S. Merenkov establishes the following properties:

Theorem 8.10. *The spaces X and $\widehat{X}$ are geodesic carpets, linearly locally connected, Ahlfors regular of dimension 2. Their conformal dimension is 2. The peripheral circles are uniform quasicircles, and are uniformly separated.*

It follows that neither X nor $\widehat{X}$ admit any quasisymmetric embedding into $\mathbb{S}^2$. Concerning their group of quasi-Möbius maps, S. Merenkov has established the following properties:

Theorem 8.11.

1. *The space X is rigid: its group of quasi-Möbius maps is the group of isometries of X, isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*
2. *The group of quasi-Möbius maps of $\widehat{X}$ coincides with the group of bi-Lipschitz homeomorphisms. It is uncountable.*

The main point is that the branched foliation coming from the vertical curves via the projection p plays a special role: only families of vertical curves may have positive 2-modulus. Therefore, any quasi-Möbius map preserves the vertical curves. For X this forces rigidity. For $\widehat{X}$, one may rotate along the vertical curves and obtain maps of the form $(x, y) \mapsto (x, y + r_x)$. Let us note that we obtain drastically different behaviors compared to round carpets.

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