DYNAMICAL SYSTEMS
BANACH CENTER PUBLICATIONS, VOLUME 115
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2018

SOME TOPOLOGICAL CHARACTERIZATIONS OF RATIONAL MAPS AND KLEINIAN GROUPS

PETER HAÏSSINSKY

Aix Marseille Univ, CNRS, Centrale Marseille, I2M
Marseille, France
E-mail: phaissin@math.cnrs.fr

Abstract. The aim of this course is to present methods coming from quasiconformal geometry in metric spaces which can be used to characterize conformal dynamical systems.

We will focus on some specific classes of rational maps and of Kleinian groups (semi-hyperbolic rational maps and convex-cocompact Kleinian groups). These classes can be characterized among conformal dynamical systems by topological properties, which will enable us to define classes of dynamical systems on the sphere (coarse expanding conformal maps and uniform convergence groups). It turns out that these topological dynamical systems carry some non-trivial geometric information enabling us to associate a coarse conformal structure invariant by their dynamics. This conformal structure will be derived from hyperbolic geometry in the sense Gromov. We associate to this conformal structure a numerical invariant, the *Ahlfors regular conformal dimension*, which will contain the information that such topological dynamical systems are conjugate to genuine conformal dynamical systems.

1. Introduction. We introduce the main objects of the course and the principal theorem (Theorem 1.3) we wish to prove.

1.1. Classical conformal dynamical systems. In the early eighties, Sullivan established a dictionary between different kinds of conformal dynamical systems [Sul3]. We focus on particular subclasses of rational maps and of Kleinian groups.

1.1.1. *Semi-hyperbolic rational maps.* References on holomorphic dynamics include [CG, Mil].

2010 *Mathematics Subject Classification*: 20F67, 30B05, 37F10, 37F30, 30L, 57M60.
Key words and phrases: hyperbolic geometry, conformal gauge, sewing, convex-cocompact Kleinian groups, semi-hyperbolic rational maps, quasisymmetric maps, convergence groups, cxc dynamics.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc115-3

Let $f = P/Q$ be a rational map, where P and Q are relatively prime polynomials in $\mathbb{C}[z]$. Its degree is by definition $\max\{\deg P, \deg Q\}$; it is the number of preimages of a point counted with multiplicity. A rational map f acts on the Riemann sphere $\widehat{\mathbb{C}}$ by iteration. A general goal is to understand the asymptotic behavior of its iterates

$$f^n = \underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}}.$$

If f has degree $d \geq 2$, then the sphere can be decomposed into two totally invariant sets

$$\widehat{\mathbb{C}} = J_f \ \sqcup \ F_f$$

where

- J_f is the Julia set, where the action is chaotic,
- F_f is the Fatou set, where the action is equicontinuous.

More precisely, F_f is the set of points z that admit a neighborhood V such that $\{f^n|_V\}_{n\geq 0}$ is equicontinuous. The Julia set is defined as its complement. It is also the closure of the set of repelling cycles.

Its critical points consist of those points at which the rational map is not locally injective, i.e., for which the derivative vanishes (in appropriate charts if the point at infinity is involved). We say that f is *semi-hyperbolic* if J_f contains no recurrent critical points and if every cycle is either attracting or repelling [CJY].

1.1.2. *Convex-cocompact Kleinian groups.* Background on Kleinian groups include [Mar, Ser]. A Kleinian group is a discrete subgroup G of $\mathrm{PSL}_2(\mathbb{C})$. It acts on the Riemann sphere $\widehat{\mathbb{C}}$ by Möbius transformations.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \rightsquigarrow \quad z \mapsto \frac{az+b}{cz+d}.$$

As for rational maps, the action of a Kleinian group G decomposes the sphere into two invariant sets:

$$\widehat{\mathbb{C}} = \Lambda_G \ \sqcup \ \Omega_G$$

where

- Λ_G is the limit set, where the action is chaotic,
- Ω_G is the ordinary set, where the action is properly discontinuous: for any compact subsets $K, L \subset \Omega_G$, the set $\{g \in G, \ g(K) \cap L \neq \emptyset\}$ is finite.

More precisely, Ω_G is the locus of points $z \in \widehat{\mathbb{C}}$ that admit a neighborhood V such that $\{g|_V\}_{g\in G}$ is equicontinuous. The limit set is defined as its complement. It is also the cluster set of the orbit of any point.

Moreover, a Kleinian group G also acts properly discontinuously on $\mathbb{H}^3$ by isometries. Identifying $\mathbb{H}^3$ with the Euclidean unit ball $\mathbb{B}^3 \subset \mathbb{R}^3$ and $\widehat{\mathbb{C}}$ with the Euclidean unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, one obtains an action of G on $\mathbb{B}^3 \cup \mathbb{S}^2$, which is properly discontinuous on $\mathbb{B}^3 \sqcup \Omega_G$.

When G is torsion-free, the quotient

$$M_G = (\mathbb{B}^3 \sqcup \Omega_G)/G$$

defines the *Kleinian manifold* of G.

Since the action of G preserves its limit set, it also preserves its convex hull $\mathrm{Hull}(\Lambda_G) \subset \mathbb{H}^3$. We say that the Kleinian group is *convex-cocompact* if $\mathrm{Hull}(\Lambda_G)/G$ is compact. Equivalently, M_G is compact [Sul1].

1.1.3. *Topological characterization.* Semi-hyperbolic rational maps and convex-cocompact Kleinian groups satisfy a *conformal elevator principle.* This means that the dynamics enables us to go from small scales to large scales with bounded distortion (this implies the self-similarity of the chaotic locus). This is expressed as follows.

- Let G be a convex-cocompact Kleinian group; there exist definite sizes $r_1 > r_0 > 0$ such that, for any $x \in \Lambda_G$, for any $r > 0$, there is an element $g \in G$ such that
$$B(g(x), r_0) \subset g(B(x,r)) \subset B(g(x), r_1).$$
- Let f be a semi-hyperbolic rational map: there exist a definite size $r_0 > 0$ and a maximal degree $d_{max} < \infty$ such that, for any $x \in J_f$, for any $r > 0$, there is an iterate $n \geq 0$ such that
$$f^n(B(x,r)) \supset B(f^n(x), r_0)$$
and the restriction $f^n|_{B(x,r)}$ is at most d_{max}-valent.

This property characterizes these classes [Sul1, CJY].

1.2. Topological counterparts. In this section, we present the main problems tackled in these lectures and we provide topological analogs for convex-cocompact Kleinian groups and semi-hyperbolic rational maps.

Central problem: topological/dynamical characterization. An important theme in hyperbolic geometry and conformal dynamics is to determine to which extent such objects can be characterized topologically. In the realm of manifolds, this is the content of uniformization theorems (uniformization of surfaces and 3-manifolds), and for rational maps, an instance is provided by Thurston's characterization of postcritically finite rational maps. In these lectures, we will be concerned by the following questions.

- When is a group of homeomorphisms of the sphere conjugate to a (convex-cocompact) Kleinian group ?
- When is a finite branched covering of the sphere conjugate to a (semi-hyperbolic) rational map ?

1.2.1. *Convergence groups.* Let G be a group of orientation-preserving homeomorphisms of the sphere. Following Gehring and Martin, G is a **convergence group** if its action on the set of distinct triples is properly discontinuous [GM, Tuk2, Bow2].

As for Kleinian groups, we have the dynamical decomposition
$$S^2 \;=\; \Lambda_G \;\sqcup\; \Omega_G.$$

The action is **uniform** on Λ_G if its action on the set of distinct triples is cocompact.

THEOREM 1.1. *A Kleinian group is convex-cocompact iff it is a convergence group, uniform on its limit set.*

To each triple of distinct points of the Riemann sphere corresponds the incenter of the ideal triangle in hyperbolic space they define: the discreteness of a Kleinian group is equivalent to the convergence property and the convex-cocompactness is equivalent to the uniform property of the action on its limit set.

1.2.2. *Topological cxc maps.* This class of maps was introduced in [HP1] in a very general setting. We focus on those which are defined on the sphere. Let $f\colon S^2 \to S^2$ be an orientation-preserving finite branched covering of the sphere of degree $d \geq 2$, and let us assume that $\mathfrak{X}_1 \subset\subset \mathfrak{X}_0 \subset S^2$ are open subsets of the sphere such that $f\colon \mathfrak{X}_1 \to \mathfrak{X}_0$ is also a finite branched covering of degree d. Let $X = \bigcap f^{-n}(\mathfrak{X}_0)$ denote the *repellor* of f.

Given a finite cover $\mathcal{U}$ of X by connected sets, one defines a sequence of covers $\{\mathcal{U}_n\}$ by letting $\mathcal{U}_n$ denote the collection of connected components of $f^{-n}(U)$ for $U \in \mathcal{U}$.

The map $f\colon \mathfrak{X}_1 \to \mathfrak{X}_0$ is *topologically coarse expanding conformal* (*top. cxc*) if there exists a finite covering $\mathcal{U}$ of X satisfying the following properties.

(1) [Irreducibility] For any open set U that intersects X, there is some iterate $n \geq 0$ such that $X \subset f^n(U)$.
(2) [Expansion]
$$\lim_{n\to\infty} \max\{\operatorname{diam} W,\ W \in \mathcal{U}_n\} = 0.$$
(3) [Degree] There is some maximal degree $d_{max} \in \mathbb{N}$ such that, for all $n \geq 1$ and $W \in \mathcal{U}_n$,
$$\deg(W \xrightarrow{f^n} f^n(W)) \leq d_{max}.$$

Theorem 1.2 ([HP1, Cor. 4.2.2]). *A rational map is top. cxc iff it is a semi-hyperbolic map.*

1.3. Topological characterizations: an analytic approach. We propose an approach based on quasiconformal geometry in metric spaces. The basic steps are the following:

(1) Find a coarse conformal structure preserved by the dynamics.
(2) Recognize the conformal structure of $\widehat{\mathbb{C}}$.

Let $\mathcal{D}$ denote either the iterates of a topological cxc map or a convergence group that has a uniform action on its limit set and let X denote either the repellor of the map or the limit set of the group.

By a conformal structure, we will mean a family of metrics $\mathcal{G}(\mathcal{D})$ on X such that $\mathcal{D}$ acts by conformal maps in a coarse sense. This roughly means that there exists a constant $H \geq 1$ such that, for any $x \in X$ and $g \in \mathcal{D}$, there is some definite size $r_0 > 0$ with the following property: for any $r \in (0, r_0)$, there is some size $s > 0$ such that
$$B(g(x), s) \subset g(B(x,r)) \subset B(g(x), Hs).$$
We will concentrate on the *Ahlfors regular conformal gauge* $\mathcal{G}_{AR}(\mathcal{D})$, i.e., those metrics of $\mathcal{G}(\mathcal{D})$ that are Ahlfors regular. A metric space X is *Ahlfors regular*, or *Ahlfors Q-regular* to be more precise, if there is a Radon measure μ such that for any $x \in X$ and $r \in (0, \operatorname{diam} X]$, $\mu(B(x,r)) \asymp r^Q$ for some given $Q > 0$[1] [Mat]. The measure μ is then equivalent to the Hausdorff measure on X of dimension Q. The *Ahlfors regular conformal*

[1] We write $a \asymp b$ for two positive quantities a and b if is there is a universal constant $u \geq 1$ such that $a/u \leq b \leq ua$.

dimension $\mathrm{confdim}_{AR}\mathcal{D}$ of $\mathcal{D}$ is defined as the infimum over $\mathcal{G}_{AR}(\mathcal{D})$ of every dimension Q [MT, CP, Haï1]. This is a numerical invariant of $\mathcal{D}$. It is a refinement due to Bourdon and Pajot [BP] of Pansu's notion of conformal dimension [Pan]. If there is a metric in $\mathcal{G}_{AR}(\mathcal{D})$ of minimal dimension, that is, of dimension $\mathrm{confdim}_{AR}\mathcal{D}$, then we say that $\mathrm{confdim}_{AR}(\mathcal{D})$ *is attained.*

We will prove that this numerical invariant suffices to characterize semi-hyperbolic rational maps and convex-cocompact Kleinian groups.

THEOREM 1.3. *Let $\mathcal{D}$ denote either the iterates of a topological cxc map or a convergence group that has a uniform action on its limit set and let X denote either the repellor of the map or the limit set of the group. Assume that X is connected. If*

(1) $X = S^2$, $\mathrm{confdim}_{AR}\mathcal{D} = 2$ *and is attained, or*
(2) $X \subsetneq S^2$ *and* $\mathrm{confdim}_{AR}\mathcal{D} < 2$

then $\mathcal{D}$ is conjugate to a genuine conformal dynamical system.

Case (1) is due to Bonk and Kleiner when $\mathcal{D}$ is a group [BK2] and Haïssinsky and Pilgrim when $\mathcal{D}$ is a map [HP2]; case (2) follows from [Haï3, Haï4]. Note that in case (1) more is known [BK3, HP2]:

THEOREM 1.4. *Let $\mathcal{D}$ denote either the iterates of a topological cxc map with repellor S^2 or a convergence group that has a uniform action on S^2 and assume that the Ahlfors regular conformal dimension is attained.*

(1) *If $\mathcal{D}$ is a group, then $\mathcal{D}$ is conjugate to a cocompact Kleinian group.*
(2) *If $\mathcal{D}$ is cxc, then $\mathcal{D}$ is conjugate to either a rational map, or to a real Lattès map (with simple real eigenvalues).*

1.4. Outline. To start with, we will show that the limit set of a convergence group or the repellor of a topological cxc map carries a canonical coarse conformal structure as above. This conformal structure is the trace at infinity of a hyperbolic space in the sense of Gromov. From this structure, we will show how to embed these sets into a metric sphere and how to extend the dynamics to this sphere in a conformal fashion. Assuming a control on the Ahlfors regular conformal dimension, we will then be able to prove that our dynamical system is conjugate to either a Kleinian group or a rational map.

The rest of the course is organized in four chapters. We will proceed backwards, and explain how to conclude the proof.

(1) **Quasiconformal geometry.** We recall the basics on quasiconformal maps and quasiconformal geometry. We will state the theorems we wish to apply to establish that coarse conformal dynamical systems are conjugate to genuine conformal dynamical systems. This will provide us with an aim to achieve. We will also explain how to construct a metric sphere from a metric compact planar set.
(2) **Hyperbolic geometry.** We introduce hyperbolic geometry in the sense of Gromov and explain how this defines a canonical coarse conformal structure on a compact set, once we are given a suitable sequence of finite open covers of a Hausdorff compact space.

(3) **Convergence actions.** In this chapter, we sketch the proof of Theorem 1.3 in the group case. This situation is easier than for cxc maps because the topology of the limit set is simpler.
(4) **Cxc dynamics.** We sketch here the proof of Theorem 1.3 in the non-invertible case. We will emphasize on the differences from the group case.

2. Quasiconformal geometry. Quasiconformal geometry consists of those properties of metric spaces that are preserved by quasiconformal maps and their variants. We first review the theory of quasiconformal maps in metric spaces, before specializing ourselves to maps on the plane. This will allow us to motivate the strategy of the proof of the main theorem. We then explain how to construct metric spheres and how to control their geometry.

2.1. Different classes of maps in metric spaces. Let (X, d_X) and (Y, d_Y) be two metric spaces and let $f\colon X \to Y$ be a homeomorphism. For any $x \in X$, any $R > 0$ set

$$\begin{aligned} L_f(x,R) &= \sup\{d_Y(f(x), f(y)) : d_X(x,y) \le R\},\\ \ell_f(x,R) &= \inf\{d_Y(f(x), f(y)) : d_X(x,y) \ge R\},\\ H_f(x,R) &= \frac{L_f(x,R)}{\ell_f(x,R)}. \end{aligned}$$

We say that f is

- *quasiconformal* (QC) if there exists $H \ge 1$ so that $\limsup_{R\to 0} H_f(x,R) \le H$ for any $x \in X$;
- *weakly quasisymmetric* (WQS) if there exists $H \ge 1$ so that $H_f(x,R) \le H$ for any $x \in X$ and any $R > 0$;
- *quasisymmetric* (QS) if there exists a homeomorphism $\eta_f = \eta\colon [0,+\infty) \to [0,+\infty)$ so that
$$d_X(x,a) \le t d_X(x,b) \Rightarrow d_Y(f(x), f(a)) \le \eta(t) d_Y(f(x), f(b))$$
whenever a, b $x \in X$ and $t > 0$;
- *quasi-Möbius* (QM) if there exists a homeomorphism $\theta_f = \theta\colon [0,+\infty) \to [0,+\infty)$ such that, for any distinct points x_1, x_2, x_3 and x_4 in X,
$$[f(x_1), f(x_2), f(x_3), f(x_4)] \le \theta([x_1, x_2, x_3, x_4]),$$
where $[x_1, x_2, x_3, x_4]$ denotes the metric crossratio defined by
$$[x_1, x_2, x_3, x_4] = \frac{d(x_1,x_2)d(x_3,x_4)}{d(x_1,x_3)d(x_2,x_4)}.$$

Roughly speaking, QC homeomorphisms (respectively WQS homeomorphisms) distort the shape of infinitesimal balls (respectively every ball) by a uniformly bounded amount. The (QS) condition is a scale-free condition, hence a priori more restrictive. The (QM) condition appears naturally when looking at group actions.

For example, any bi-Lipschitz homeomorphism (with constant L) is QS (and in this case $\eta(t) = L^2 t$). It is also easy to see that the inverse of a QS homeomorphism is QS ($\eta_{f^{-1}}(t) = 1/\eta^{-1}(t^{-1})$) and that the composition of two QS homeomorphisms is QS

($\eta_{f\circ g} = \eta_f \circ \eta_g$). Note that these properties are not obvious for QC nor WQS homeomorphisms.

The basic distortion bound for quasisymmetric maps is given by the following lemma [Hei, Prop. 10.8].

LEMMA 2.1. *Let $h\colon X \to Y$ be an η-quasisymmetric map between compact metric spaces. For all $A, B \subset X$ with $A \subset B$ and $\operatorname{diam} B < \infty$, we have $\operatorname{diam} h(B) < \infty$ and*

$$\frac{1}{2\eta\left(\frac{\operatorname{diam} B}{\operatorname{diam} A}\right)} \leq \frac{\operatorname{diam}\, h(A)}{\operatorname{diam} h(B)} \leq \eta\left(2\frac{\operatorname{diam} A}{\operatorname{diam} B}\right).$$

Another instance is the following precompactness result [TV, Thm 3.4].

THEOREM 2.2. *Let $\eta\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a fixed distortion function, $M < \infty$ and X, Y be two metric spaces, $x, x' \in X$. Then*

$$\{f\colon X \to Y \ \eta\text{-quasisymmetric such that } d(f(x), f(x')) \leq M\}$$

is equicontinuous. Furthermore, any limit of such functions is either constant or η-quasisymmetric.

All these notions of coarse conformality are related as follows.

THEOREM 2.3. *Let (X, d_X) and (Y, d_Y) be two metric spaces. Denote by (QC), (QS) and (WQS) the class of injective maps from X to Y that are respectively quasiconformal, quasisymmetric and weakly quasisymmetric. The following assertions are true.*

(i) *The following implications hold: (QS) $\Rightarrow$ (WQS) $\Rightarrow$ (QC).*
(ii) *If X is doubling and connected and Y is doubling, then (WQS) $\Rightarrow$ (QS).*

A metric space Z is *doubling* if there exists an integer N such that any set of finite diameter can be covered by at most N sets of half its diameter.

We have the following relationships with quasi-Möbius maps:

THEOREM 2.4. *The relationships between quasi-Möbius maps and quasisymmetric maps are as follows.*

(i) *A quasi-Möbius is uniformly locally quasisymmetric.*
(ii) *A quasisymmetric map is quasi-Möbius.*
(iii) *Let $f\colon X \to Y$ be a quasi-Möbius map between metric spaces. If X and Y are unbounded, then f is quasisymmetric if and only if $f(x)$ tends to infinity when x tends to infinity. If X and Y are bounded, and if for three points $z_1, z_2, z_3 \in X$, we have $|z_i - z_j| \geq \operatorname{diam} X/\lambda$ and $|f(z_i) - f(z_j)| \geq \operatorname{diam} Y/\lambda$ for some $\lambda > 0$, then f is η-quasisymmetric, where η only depends λ and on the distortion of crossratios.*

Notes. The literature on quasiconformal maps is plentiful. The classical references include [Ahl, LV, Väi1]; see also [AIM]. The definition of quasiconformal mappings given here has been introduced by F. Gehring [Geh]. The notion of quasisymmetry is due to P. Tukia and J. Väisälä [TV] while the notion of weak quasisymmetry goes back to A. Beurling and L. Ahlfors [BA]. The theory of quasi-Möbius maps was developed by J. Väisälä [Väi2].

2.2. The planar case and the measurable Riemann mapping theorem. We specify the above discussion to the complex plane, where such maps were actually introduced.

Let U be an open set of the Riemann sphere $\widehat{\mathbb{C}}$ and $f\colon U \to \widehat{\mathbb{C}}$ be a continuous, not necessarily invertible, non-constant map. The map f is called *quasiregular* provided f belongs to the Sobolev space $W^{1,2}_{loc}(U)$, and, for some $K < \infty$, satisfies

$$|Df(x)|^2 \le K \cdot J_f(x) \quad a.e. \tag{2.1}$$

where $|Df(x)|$ is the spherical operator norm of the derivative and $J_f(x)$ is the Jacobian determinant. In this case we also say that f is K-*quasiregular*. If in addition $f\colon U \to f(U)$ is a homeomorphism, f is said to be K-*quasiconformal* (in the analytic sense).

A quasiregular map is discrete and open. The *branch set* B_f is the set of points at which f fails to be a local homeomorphism. The branch set B_f and its image $f(B_f)$ have measure zero. Moreover, f is differentiable almost everywhere and the Jacobian is positive almost everywhere. Condition (2.1) implies that at almost every point x in $U \setminus B_f$, the derivative sends round balls to ellipsoids of uniformly bounded eccentricity. The composition of a K-quasiregular map with a conformal map is again K-quasiregular. The inverse of a quasiconformal map is quasiconformal, and the composition of quasiregular maps is quasiregular.

One characterization due to S. Kallunki and P. Koskela [KK, KR] is given in terms of the asymptotic pointwise distortion of the roundness of balls as defined below.

Roundness. Let A be a bounded, proper subset of a metric space X with non-empty interior. Given an interior point $a \in \mathrm{int}(A)$, define the *outradius* of A about a as

$$L(A, a) = \sup\{|a - b| : b \in A\}$$

and the *inradius* of A about a as

$$\ell(A, a) = \sup\{r : r \le L(A, a) \ \text{ and } \ B(a, r) \subset A\}.$$

The *roundness of A about a* is defined as

$$\mathrm{Round}(A, a) = L(A, a)/\ell(A, a) \in [1, \infty).$$

Theorem 2.5. *A non-constant mapping $f\colon U \to \widehat{\mathbb{C}}$ is quasiregular if and only if*

(1) *f is continuous, orientation-preserving, discrete, and open;*
(2) *for every point $x \in U$ outside a countable set, there exists a basis of open neighborhoods (U_n) such that*

$$\limsup_{n\to\infty} \max\{\mathrm{Round}(U_n, x), \mathrm{Round}(f(U_n), f(x))\} < \infty$$

and there exists a constant $H < \infty$ such that

$$\limsup_{n\to\infty} \max\{\mathrm{Round}(U_n, x), \mathrm{Round}(f(U_n), f(x))\} \le H$$

almost everywhere.

In particular, the following conditions are equivalent for an orientation-preserving homeomorphism $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$:

(1) *f is quasiconformal in the analytic sense;*
(2) *f is quasiconformal in the metric sense;*

(3) *f is quasisymmetric;*
(4) *f is quasi-Möbius.*

A variant of Stoïlow's factorisation theorem implies that a quasiregular map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is the composition of a rational map and a quasiconformal map.

In complex notation, Condition (2.1) reads

$$\left|\frac{\partial f}{\partial \bar z}\right| \le \frac{K-1}{K+1}\left|\frac{\partial f}{\partial z}\right| \quad \text{a.e.}$$

Since the Jacobian is positive almost everywhere, it follows that $|\partial f/\partial z| > 0$ almost everywhere as well. Therefore, we may associate its Beltrami coefficient

$$\mu_f = \frac{\partial f/\partial \bar z}{\partial f/\partial z}$$

which defines an element in the open unit ball of L^∞.

The measurable Riemann mapping theorem, specific to dimension 2, implies the converse:

Theorem 2.6 (Measurable Riemann mapping theorem). *Let $U \subset \widehat{\mathbb{C}}$ be an open set and let $\mu \in L^\infty(U)$ with $\|\mu\|_\infty < 1$. There exists a quasiconformal mapping $f\colon U \to f(U)$ such that $\mu = \mu_f$ almost everywhere.*

If $g\colon U \to g(U)$ is an element of $W^{1,2}_{loc}$ such that

$$\frac{\partial g}{\partial \bar z} = \mu \frac{\partial g}{\partial z} \quad \textit{a.e.},$$

then there exists a holomorphic map $h\colon f(U) \to g(U)$ such that $g = h \circ f$.

2.3. Geometric characterizations of conformal dynamical systems. The purpose of this section is the following theorem due to Sullivan which is the basic result of surgery.

Theorem 2.7. *Let $\mathcal{D}$ be a countable group of quasi-Möbius homeomorphisms of the sphere or the iterates of a quasiregular map. If there is a constant K such that g is K-quasiregular for all $g \in \mathcal{D}$, then $\mathcal{D}$ is quasiconformally conjugate to a collection of rational maps.*

Its proof is oultined in [Sul2, Sul4]. We sketch P. Tukia's approach [Tuk1] adapted by A. Hinkkanen [Hin]. The main idea is to find an invariant Beltrami form under $\mathcal{D}$ and to apply the measurable Riemann mapping theorem. A Beltrami form can be seen as a measurable section of $\widehat{\mathbb{C}}$ to the bundle of conformal structures, which are identified with the Poincaré disk. The action of a quasiregular map is a hyperbolic isometry between fibers. Thus, since the action of $\mathcal{D}$ is uniformly quasiregular, we may find a bounded set on almost each fiber that is invariant under $\mathcal{D}$. Since the curvature of $\mathbb{H}^2$ is non-positive, such sets are contained in unique hyperbolic disks of minimal radius. These disks are invariant under $\mathcal{D}$ and their centers define the invariant Beltrami form that was looked for.

Remark 2.8. Theorem 2.7 does not hold for arbitrary semigroups, as was shown by A. Hinkkanen. The simplest counterexample is certainly the following. Let h be a quasiconformal map with its support in $(0,1) \times i\mathbb{R}$ and set $f_1(z) = z+1$ and $f_2 = f_1 \circ h$. Then the semigroup generated by $\{f_1, f_2\}$ is uniformly quasiconformal but not conjugate to a semigroup of Möbius transformations as soon as h is not conformal.

Conformal gauge revisited. Let (X, d_X) be a metric space. Its *conformal gauge* $\mathcal{G}(X, d_X)$ is the set of metrics d on X such that the identity map $\mathrm{id}\colon (X, d_X) \to (X, d)$ is quasisymmetric [Hei, Chap. 15]. Its *Ahlfors regular conformal gauge* $\mathcal{G}_{AR}(X)$ is the subset of metrics of $\mathcal{G}(X)$ that admit an Ahlfors regular measure (such a space may be empty; it is not if X is connected and doubling). We may then define the *conformal dimension* $\mathrm{confdim}\, X$ and the *Ahlfors regular conformal dimension* $\mathrm{confdim}_{AR} X$ as the infimum of the Hausdorff dimensions of (X, d) over $d \in \mathcal{G}(X)$ and $d \in \mathcal{G}_{AR}(X)$ respectively. The following relationships generally hold

$$\dim_{top} X \leq \mathrm{confdim}\, X \leq \mathrm{confdim}_{AR} X.$$

Examples with strict inequalities exist. The description of the conformal gauge of a compact metric space is done in [CP]; more information on conformal dimension can be found in [MT].

We go one step further to characterizing conformal dynamical systems among maps acting on metric spaces by first providing a characterization of the Riemann sphere up to quasisymmetry.

THEOREM 2.9 (Bonk and Kleiner [BK1]). *A metric 2-sphere is quasisymmetrically equivalent to the Riemann sphere if it is linearly locally connected and Ahlfors 2-regular.*

A metric space Z is *linearly locally connected* if there is a constant $\lambda \geq 1$ such that, for all $z \in Z$ and $R > 0$,

(LLC1) for all $x, y \in B(z, R)$ there is a continuum $E \subset B(z, \lambda R)$ that contains $\{x, y\}$;
(LLC2) for all $x, y \notin B(z, R)$, there is a continuum $E \subset Z \setminus B(z, (1/\lambda)R)$ that contains $\{x, y\}$.

Combining Theorems 2.3, 2.9 and 2.7 provides us with the following corollary.

COROLLARY 2.10. *Let X be an LLC metric sphere, $H > 0$ a constant and $\mathcal{D}$ a group or the semi-group induced by a map acting on X. Let us assume that, for all $g \in \mathcal{D}$, for all $x \in X$, there is a neighborhood V such that $g|_V$ is H-weakly quasisymmetric. If $\mathcal{G}_{AR}(X)$ contains an Ahlfors 2-regular metric, then there exists a quasisymmetric map $f\colon X \to \widehat{\mathbb{C}}$ such that $f \circ \mathcal{D} \circ f^{-1}$ is a collection of rational maps.*

2.4. Construction of metric spheres. The purpose of this section is to explain how to define a metric on a sphere from a collection of metric planar continua which are pieced together. The general setting is a collection of metric spaces $(X_\alpha, d_\alpha)_{\alpha \in A}$ together with gluing instructions which can be described by an equivalence relation on the disjoint union $Y = \bigsqcup X_\alpha$. The most natural way is to define, for points $x, y \in Y$,

$$\delta(x, y) \overset{\text{def.}}{=} \inf \sum_{j=1}^{n} d_{\alpha_j}(x_j, y_j) \tag{2.2}$$

where the infimum is taken over all finite chains $\{x_j, y_j\}_{1 \leq j \leq n}$ such that $x_1 = x$, $y_n = y$, x_j, y_j belong to the same space X_{α_j} for each index j and y_j is equivalent to x_{j+1} for

$j = 1, \ldots, n-1$. If no such chain exists, one sets $\delta(x, y) = \infty$. This provides us with a semi-metric which takes its values in $[0, \infty]$; cf. [BBI, Chapter 3] for a general discussion on this topic.

We are interested in a much more constrained setting that we describe now. We assume that $X \subset \mathbb{S}^2$ is a compact, connected and locally connected subset of $\mathbb{S}^2$. Let us denote by $\mathcal{C}(X)$ the collection of components of $\mathbb{S}^2 \setminus X$. Each one is simply connected, and, for any $\delta > 0$, the collection of components of diameter at least δ is finite, and the boundary of each component is locally connected according to [Why, Thm. VI.4.4]. Thus, if $\Omega \in \mathcal{C}(X)$, there exists a surjective continuous map $\varphi_\Omega : \overline{\mathbb{D}} \to \overline{\Omega}$ that is one-to-one on $\mathbb{D}$, with totally disconnected fibers. These maps give rise to the equivalence relation generated by $\{(x, \varphi_\Omega(x)) : x \in \mathbb{S}^1, \Omega \in \mathcal{C}(X)\}$ on the disjoint union of X with a collection of closed disks $\overline{\mathbb{D}}$ in bijection with $\mathcal{C}(X)$.

We make the following assumptions.

- The space X is endowed with a metric d_X.
- For every element $\Omega \in \mathcal{C}(X)$, we assume that we are given a metric d_Ω on $\overline{\mathbb{D}}$ such that $\varphi_\Omega : (\mathbb{S}^1, d_\Omega) \to (\partial\Omega, d_X)$ is 1-Lipschitz and that there is a constant $\Delta > 0$ such that $\operatorname{diam}_{d_\Omega} \overline{\mathbb{D}} \le \Delta \operatorname{diam}_{d_X} \partial\Omega$.

This enables us to replace the fornula (2.2) by the following two-step construction. We first define a metric on the closure of each Ω, $\Omega \in \mathcal{C}(X)$.

FACT 2.11. *Define $m_\Omega : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}_+$ as follows:*

– *If $y, y' \in \partial\Omega$, let $m_\Omega(y, y') = d_X(y, y')$.*
– *If $z \in \Omega$ and $y \in \partial\Omega$, let*

$$m_\Omega(z, y) = m_\Omega(y, z) = \inf\{d_\Omega(\varphi_\Omega^{-1}(z), x) + d_X(\varphi_\Omega(x), y) : x \in \mathbb{S}^1\}.$$

– *If $z, z' \in \Omega$, let*

$$\begin{aligned} m_\Omega(z, z') = \min\{&d_\Omega(\varphi_\Omega^{-1}(z), \varphi_\Omega^{-1}(z'));\\ &\inf\{d_\Omega(\varphi_\Omega^{-1}(z), x) + d_X(\varphi_\Omega(x), \varphi_\Omega(x')) + d_\Omega(x', \varphi_\Omega^{-1}(z')) : x, x' \in \mathbb{S}^1\}\}. \end{aligned}$$

Then $(\overline{\Omega}, m_\Omega)$ is a metric space, the map $\varphi_\Omega : \overline{\mathbb{D}} \to \overline{\Omega}$ becomes a 1-Lipschitz map, the restriction $\varphi_\Omega : B(z, d_\Omega(z, \mathbb{S}^1)/2) \to B(\varphi_\Omega(z), m_\Omega(\varphi_\Omega(z), \partial\Omega)/2)$ is an isometry for all $z \in \mathbb{D}$ and $m_\Omega(x, \partial\Omega) = d_\Omega(\varphi_\Omega^{-1}(x), \mathbb{S}^1)$ for any $x \in \Omega$.

Patchwork metric. We may then define the *patchwork metric* d on S^2 as follows: let $\Omega, \Omega' \in \mathcal{C}(X)$ be two distinct components of $S^2 \setminus X$; set, given $z, w \in \Omega$, $z' \in \Omega'$, and $y, y' \in X$,

$$\begin{cases} d(y, y') = d_X(y, y'); \\ d(z, w) = m_\Omega(z, w); \\ d(z, z') = \inf\{m_\Omega(z, x) + d_X(x, x') + m_{\Omega'}(x', z') : \ x \in \partial\Omega, x' \in \partial\Omega'\}; \\ d(z, y) = d(y, z) = \inf\{m_\Omega(z, x) + d_X(x, y) : x \in \partial\Omega\}. \end{cases}$$

One may check that the identity map $\mathrm{id} : (S^2, d) \to S^2$ is a homeomorphism since $\operatorname{diam}_{d_\Omega} \overline{\mathbb{D}} \le \Delta \operatorname{diam}_{d_X} \partial\Omega$.

2.5. Geometric properties. We keep the same notation as above. The collection $\mathcal{P} = \{\partial\Omega,\ \Omega \in \mathcal{C}(X)\}$ denotes the *boundary components* of X.

We now define regular maps in the sense of David and Semmes [DS, Chap. 12], which will be used to establish geometric properties inherited by the patchwork metric.

Definition 2.12 (regular maps). Let X, Y be metric spaces and $p\colon X \to Y$ be a continuous map. Given $N, L > 0$, we say that p is (L, N)-*regular* if p is L-Lipschitz and, for any $y \in Y$ and $r > 0$, the preimage $p^{-1}(B(y, r))$ can be covered by at most N balls of radius Lr.

We note that such maps have bounded multiplicity hence are light. From now on, we will always assume that they are surjective. Moreover, if $Z \subset X$ is connected then $\operatorname{diam} Z \asymp \operatorname{diam} p(Z)$.

The main result of this section is the following (the terminology is given later on):

Theorem 2.13. *Let $X \subset S^2$ be a compact, connected and locally connected subset endowed with a metric d_X and let us assume that there are constants (L, N) such that, for every component $\Omega \in \mathcal{C}(X)$, we are given a metric d_Ω on $\overline{\mathbb{D}}$ and an (L, N)-regular 1-Lipschitz map $\varphi_\Omega\colon (\mathbb{S}^1, d_\Omega) \to (\partial\Omega, d_X)$. We make the following additional assumptions.*

- *(X, d_X) satisfies the (BT)-property, is Ahlfors Q-regular for some $Q < 2$, relatively doubling and porous with respect to its boundary components.*
- *The spaces $(\overline{\mathbb{D}}, d_\Omega)$ are uniformly Ahlfors 2-regular, LLC, the subspaces $(\mathbb{S}^1, d_\Omega)$ are uniformly porous in $(\overline{\mathbb{D}}, d_\Omega)$ and there is a constant $\Delta > 0$ such that $\operatorname{diam}_{d_\Omega} \overline{\mathbb{D}} \le \Delta \operatorname{diam}_{d_\Omega} \mathbb{S}^1$.*

If the patchwork sphere (S^2, d) is λ-LLC at every point of $X \subset (S^2, d)$ for some $\lambda \ge 1$, then (S^2, d) is quasisymmetric to $\widehat{\mathbb{C}}$.

Ahlfors regularity and the LLC property have already been defined previously. We say that a metric space Z is λ-LLC at a point z if both conditions (LLC1) and (LLC2) hold at the point z with constant λ. A metric space Z satisfies the *bounded turning property (BT)* if there is a constant $C \ge 1$ such that any pair of points $\{x, y\}$ in Z are contained in a continuum $L \subset X$ such that $\operatorname{diam} L \le Cd(x, y)$. Let us note that if X is LLC, then the patchwork sphere is LLC at every point of (S^2, d).

A metric space Z is *doubling* if there exists an integer N such that any set of finite diameter can be covered by at most N sets of half its diameter. This implies that, for all $\varepsilon > 0$, there exists N_ε such that any set E of finite diameter can be covered by N_ε sets of diameter bounded by $\varepsilon \operatorname{diam} E$. We propose a relative notion of doubling:

Definition 2.14 (Relative doubling condition). Let X be a metric continuum embedded in S^2 with boundary components $\mathcal{P}$. Then X is *doubling relative to* $\mathcal{P}$ if, for any $\varepsilon > 0$, there is some N_ε such that, for any $x \in X$ and $r > 0$, there are at most N_ε components $K \in \mathcal{P}$ such that $B(x, r) \cap K \ne \emptyset$ and $\operatorname{diam}(K \cap B(x, r)) \ge \varepsilon r$.

A subset Y of a metric space Z is said to be *porous* if there exists a constant $p > 0$ such that any ball centered at a point of Y of radius $r \in (0, \operatorname{diam} Z]$ contains a ball of radius pr disjoint from Y. We propose a relative notion of porosity:

Definition 2.15 (Relative porosity). Let X be a metric continuum embedded in S^2 with boundary components $\mathcal{P}$. Then X is *porous relative to* $\mathcal{P}$ if there exist a constant $p_X > 0$ and a maximal size $r_0 > 0$ such that, for any $x \in X$ and $r \in (0, r_0)$, there is at least one subcontinuum K' of a boundary component $K \in \mathcal{P}$ such that $K' \subset B_X(x,r)$, $K' \cap B_X(x,r/2) \neq \emptyset$ and $\operatorname{diam}_X K' \geq p_X r$.

Let us sketch the proof of Theorem 2.13. The idea is to check the assumptions of Theorem 2.9. The regularity of $\varphi_\Omega|_{\mathbb{S}^1}$ will force the global regularity of φ_Ω. From this, the bounded turning property follows. For the LLC property of the patchwork sphere, we combine the local LLC property of the domains Ω—which follows from the regularity of φ_Ω—together with the LLC at the points from X. Now, let us explain where the Ahlfors 2-regularity comes from. We need to control the 2-Hausdorff measure $\mathcal{H}^2$. The relative doubling and porosity assumptions with their uniform counterparts in X and in the different disks imply that the patchwork sphere is doubling, and that the embedded $X \hookrightarrow S^2$ is porous. Since $Q < 2$, all the mass is concentrated on $S^2 \setminus X$, which is locally uniformly Ahlfors 2-regular by construction. The doubling condition and the diameter bounds imply that there are not too many holes: this observation enables us to obtain controlled upper bounds on the mass of balls. For the lower bound, we use the porosity to ensure that there is at least one ball in some component Ω with definite mass.

2.6. A word on regular maps. Regularity enables to pull-back the (BT) property quantitatively under injectivity assumptions:

Lemma 2.16. *Let $f\colon X \to Y$ be a regular map between metric spaces X and Y. Let $L \subset X$ be connected, and assume that $f\colon L \to f(L) = K$ is injective and that K is λ-(BT). Then L satisfies the (BT) property and $f\colon L \to K$ is bi-Lipschitz quantitatively.*

Controlling the bounded turning property under regular maps can be an issue. There are examples of surjective regular maps $f\colon A \to B$ between compact connected metric spaces with the following properties:

(1) The space B is a (BT)-space, but A is not locally connected.
(2) The map f is injective but not bi-Lipschitz.
(3) The space A is a (BT)-space, but B is not, even if f is injective.
(4) The space B is a (BT)-space, but A is not, even if it is locally connected.

3. Hyperbolic geometry. Background on hyperbolic metric spaces include [Gro, GdlH, BH, KB].

Let X be a metric space. It is *geodesic* if any pair of points $\{x,y\}$ can be joined by a (geodesic) segment, i.e., a map $\gamma\colon [0, d(x,y)] \to X$ such that $\gamma(0) = x$, $\gamma(d(x,y)) = y$ and $d(\gamma(s), \gamma(t)) = |t - s|$ for all $s, t \in [0, d(x,y)]$. The metric space X is *proper* if closed balls of finite radius are compact.

A *triangle* Δ in a metric space X is given by three points $\{x,y,z\}$ and three segments joining them two by two. Given a constant $\delta \geq 0$, the triangle Δ is *δ-thin* if any side of the triangle is contained in the δ-neighborhood of the two others.

Definition 3.1 (Hyperbolic spaces and groups). A geodesic metric space is *hyperbolic* if there exists $\delta \geq 0$ such that every triangle is δ-thin. A group G is *word hyperbolic* if it acts geometrically on a proper, geodesic hyperbolic metric space.

A *geometric action* of a group on a metric space is an action by isometries that is properly discontinuous and cocompact. Basic examples of hyperbolic spaces are the simply connected Riemannian manifolds $\mathbb{H}^n$ of sectional curvature (-1) and $\mathbb{R}$-trees. In particular since the action of a convex-compact Kleinian group G is cocompact on $\operatorname{Hull}\Lambda_G$, it is a word hyperbolic group.

Given any finitely generated group G with a finite generating set S, we define the *Cayley graph* $\mathcal{C}(G,S)$ as the graph whose vertices are given by all the elements of the group G and edges are of the form (g, gs), $g \in G$ and $s \in (S \cup S^{-1})$. We endow $\mathcal{C}(G,S)$ with the length metric so that each edge is isometric to the segment $[0,1]$. The action of G on itself by left-translation gives rise to a geometric action of G on $\mathcal{C}(G,S)$.

A *quasi-isometry* between metric spaces X and Y is a map $\varphi\colon X \to Y$ such that there are constants $\lambda > 1$ and $c > 0$ such that:

- (quasi-isometric embedding) for all $x, x' \in X$,
$$\frac{1}{\lambda} d_X(x,x') - c \leq d_Y(\varphi(x), \varphi(x')) \leq \lambda d_X(x,y) + c,$$
- the c-neighborhood of the image $f(X)$ covers Y.

This defines morally an equivalence relation on metric spaces.[2]. Note that any two locally finite Cayley graphs of the same group are quasi-isometric. This enables us to discuss the quasi-isometry class of a finitely generated group (through the class of its locally finite Cayley graphs). More generally, Švarc-Milnor's lemma asserts that there is only one geometric action of a group on a proper geodesic metric space up to quasi-isometry [GdlH, Prop. 3.19].

3.1. Basic properties. We briefly review properties of hyperbolic geodesic spaces relevant to these lectures.

Approximation by trees. Let $k \geq 1$ and Z be the union of k segments or rays containing a common point w. There is a $(1,c)$-quasi-isometry of Z into a tree T where c only depends on δ and k. For a proof, see for instance [GdlH, Theorem 2.12]

Shadowing lemma. A *quasigeodesic* is the image of an interval by a quasi-isometric embedding; we speak of a (λ, c)-quasigeodesic if the constants of the quasi-isometric embedding are λ and c. The shadowing lemma asserts that, given δ, λ and c, there is a constant $H = H(\delta, \lambda, c)$ such that, for any (λ, c)-quasigeodesic q in a proper geodesic δ-hyperbolic metric space X, there is a geodesic γ at Hausdorff distance at most H. A proof can be found in [GdlH, Chap. 5].

It follows from the shadowing lemma that, among geodesic metric spaces, hyperbolicity is invariant under quasi-isometry: if X, Y are two quasi-isometric geodesic metric spaces, then X is hyperbolic if and only if Y is hyperbolic.

[2] To be correct, we should restrict ourselves to separable metric spaces or work with classes instead of sets.

Compactification. A proper geodesic hyperbolic space X admits a canonical compactification $X \sqcup \partial X$ at infinity, see the survey [KB]. This compactification can be defined by looking at the set of rays, i.e., isometric embeddings $r\colon \mathbb{R}_+ \to X$, up to bounded Hausdorff distance. The topology is induced by the uniform convergence on compact subsets of $\mathbb{R}_+$. The boundary can be endowed with a family of *visual distances* d_v compatible with its topology, i.e., that satisfy

$$d_v(a,b) \asymp e^{-\varepsilon d(w,(a,b))}$$

where $w \in X$ is any choice of a base point, $\varepsilon > 0$ is a *visual parameter* chosen small enough, and (a,b) is any geodesic asymptotic to rays defining a and b.

If $\Phi\colon X \to Y$ is a quasi-isometry between two geodesic hyperbolic spaces, then the shadowing lemma implies that Φ induces a homeomorphism $\phi\colon \partial X \to \partial Y$. This means that a word hyperbolic group G admits a topological boundary ∂G defined by considering the boundary of any proper geodesic metric space on which G acts geometrically.

In the case of a convex-compact Kleinian group K, a model for the boundary ∂K is given by its limit set Λ_K.

3.2. Analytic aspects. A general principle asserts that a geodesic hyperbolic group is determined by its boundary. More precisely, Paulin proved that the quasi-isometry class of a word hyperbolic group is determined by its boundary equipped with its quasiconformal structure [Pau]. This was later generalized by Bonk and Schramm to a broader context [BS].

The boundary ∂X of a proper geodesic metric space X is endowed with the conformal gauge defined by visual distances.

Quasi-isometries provide natural examples of quasi-Möbius maps:

THEOREM 3.2 ([Pau, Prop. 4.5]). *A (λ,c)-quasi-isometry between proper, geodesic, metric spaces extends as a θ-quasi-Möbius map between their boundaries, where θ only depends on λ, c, the hyperbolicity constants and the visual parameters.*

A pointed geodesic metric space (X,w) is *quasi-starlike* if there is some constant K such that any point $x \in X$ lies at distance at most K of a ray emanating from w.

Bonk and Schramm's result reads

THEOREM 3.3 (Bonk and Schramm [BS]). *Two proper quasi-starlike geodesic hyperbolic metric spaces are quasi-isometric if and only if there is quasi-Möbius homeomorphism between their boundaries.*

As a byproduct, one obtains:

THEOREM 3.4 (Paulin [Pau]). *Two non-elementary word hyperbolic groups are quasi-isometric if and only if there is quasi-Möbius homeomorphism between their boundaries.*

This implies that the conformal gauge of the boundary at infinity is a quasi-isometric invariant of a quasi-starlike geodesic, proper, hyperbolic space.

3.3. Geometrization of compact spaces. We provide a criterion for hyperbolicity of a metric space following Bowditch [Bow3, Prop. 3.1] and apply it to define a metric on a compact Hausdorff space endowed with a sequence of covers. This construction is a model for the dynamical settings.

We assume that X is endowed with a sequence of finite covers $(\mathcal{U}_n)_{n\geq 0}$ by non-empty open sets that forms a basis for the topology of X. We define a graph Γ with vertex set $\bigcup_n \mathcal{U}_n \cup \{X\}$. For a vertex W, let $|W| = n$ if $W \in \mathcal{U}_n$. An edge (W, W') is a pair of vertices such that $||W| - |W'|| \leq 1$ and $W \cap W' \neq \emptyset$.

We wish to add conditions to this sequence to ensure that Γ is hyperbolic and that its boundary is homeomorphic to X with some control on its geometry. Let $k_0 \geq 1$ and let us consider the following properties.

(C1) For any n and $k \geq k_0$, if $W, W' \in \mathcal{U}_{n+k}$ intersect, then we may find $V \in \mathcal{U}_n$ which contains $W \cup W'$;

(C2) For every $n \geq 1$, for every $W \in \mathcal{U}_n$, there is some $1 \leq k \leq k_0$, $V, V' \in \mathcal{U}_{n+k}$, such that $V \cap V' = \emptyset$ and $V \cup V' \subset W$.

(C3) For every $n \geq 1$, every $W \in \mathcal{U}_n$ contains some $W' \in \mathcal{U}_{n+k}$, $1 \leq k \leq k_0$, such that, whenever $V \in \mathcal{U}_{n+k}$ intersects $X \setminus W$, then $V \cap W' = \emptyset$.

(C4) For any $n \geq 0$, an element $W \in \mathcal{U}_n$ can be covered by at most k_0 elements of $\mathcal{U}_{n+1}$.

Theorem 3.5. *If* $(\mathcal{U}_n)$ *satisfies the properties* (C1)–(C3) *then* Γ *is hyperbolic and its boundary is homeomorphic to* X*. Moreover, if* d_v *is a visual distance of parameter* $\varepsilon > 0$*, there exists* $C \geq 1$ *such that, for any* $n \geq 1$ *and any* $W \in \mathcal{U}_n$*, there is a point* $x \in W$ *such that*

$$B\left(x, \frac{1}{C}e^{-\varepsilon n}\right) \subset W \subset B(x, Ce^{-\varepsilon n}).$$

If (C4) *holds, then* ∂X *is doubling.*

Condition (C1) will ensure that the graph is hyperbolic. Conditions (C2) and (C3) are used to control the boundary of the graph and the shape and size of the elements of the different coverings with respect to a visual distance. The criterion for hyperbolicity is the following:

Theorem 3.6 (Bowditch [Bow3]). *Let* X *be a geodesic metric space. We assume the existence of a constant* $h \geq 0$ *and of an assignment of a connected set* $L(x,y)$ *for each couple of distinct points that enjoy the following properties.*

(1) $L(x,y) = L(y,x)$ *holds for each* $x, y \in X$.

(2) $L(x,y)$ *is contained in the* h*-neighborhood of* $L(x,z) \cup L(z,y)$ *for all distinct* x, y, z.

(3) *If* $d(x,y) \leq 1$*, then* $\operatorname{diam} L(x,y) \leq h$.

Then X *is hyperbolic and there is a constant* $H < \infty$ *such that* $d_H(L(x,y), [x,y]) \leq H$ *for all* $x, y \in X$.

4. Convergence actions. This notion was first introduced by Gehring and Martin on spheres [GM]. Let Z be a compact metrizable space with at least three points. The action of a group G is a *convergence action* if its diagonal action on the set of triples is properly discontinuous. Equivalently, given any infinite sequence $(g_n)_n$, we may find a subsequence (n_k) and points $a, b \in Z$ such that $(g_{n_k})_k$ tends uniformly on compact subsets of $Z \setminus \{a\}$ to b. Such a sequence is called a *collapsing sequence.*

The action is *uniform* if its action is also cocompact on the set of distinct triples. As for Kleinian groups, the *limit set* Λ_G is by definition the unique minimal closed invariant subset of Z. It is empty if G is finite. These properties characterize word hyperbolic groups and their boundaries:

THEOREM 4.1 (Bowditch [Bow1]). *Let G be a convergence group acting on a perfect metrizable space X. The action of G is uniform on Λ_G if and only if G is word hyperbolic and there exists an equivariant homeomorphism between Λ_G and the boundary at infinity ∂G of G.*

The original proof consists in establishing that the space of triples is a coarse hyperbolic space on which G acts geometrically. In [GP], the authors provide an alternative proof closer to the point of view developed in Theorem 3.5; see also [Ger1, Ger2].

COROLLARY 4.2. *Let G be a uniform convergence group acting on Z. The following statements are true.*

(1) *The group G is finitely generated.*
(2) *Each non-trivial connected component of Z is locally connected with no cut point* [Swa].
(3) *There is a natural conformal gauge on Λ_G such that G acts by uniform quasi-Möbius maps.*

By Theorem 4.3 below, the metrics of this conformal gauge are exactly those metrics compatible with the topology of ∂G for which the action of G is *uniformly quasi-Möbius*, meaning that the distortion control θ is independent of $g \in G$. Visual distances are examples of metrics of the gauge. For the sake of simplicity, we will always assume that G is one-ended and torsion-free. This will in particular imply that ∂G is connected [GdlH, Prop. 7.17].

THEOREM 4.3. *Let X and X' be two perfect compact metric spaces. We are given a group G that acts on X and X' as a uniform convergence group by uniformly quasi-Möbius maps.*

(1) *Any conjugacy between the actions is quasisymmetric.*
(2) *The spaces X and X' are quasisymmetrically equivalent.*

A similar result appears in [Tuk2].

4.1. Topological facts on convergence actions on the sphere. We assume throughout this section that G is a convergence group on S^2, uniform on its limit set, torsion-free and one-ended. Properties of the limit set are first given and then we focus on the ordinary set.

The lack of cutpoints provides us with the following properties of limit sets.

PROPOSITION 4.4 ([Haï3, Prop. 5.10]). *Let G be a one-ended convergence group acting on S^2, uniform on its limit set. The limit set is connected and locally connected. Either ∂G is homeomorphic to the sphere or each component of $S^2 \setminus \partial G$ is a Jordan domain.*

We now turn to the ordinary set. The cocompact action on the set of triples provides us with this Ahlfors finiteness property (cf. [Haï3, Prop. 5.2 and 5.11]).

Proposition 4.5. *Let G be a one-ended torsion-free convergence group, with a uniform action on its limit set. Then*

(1) *the quotient Ω_G/G is a finite union of closed connected surfaces.*
(2) *the stabilizer of each component is isomorphic to a cocompact Fuchsian group.*

4.2. Geometry behind uniform convergence actions. Let G be a one-ended, convergence group acting on S^2 whose action is uniform on its limit set. We fix a metric d_Λ on Λ_G from the gauge of G. The geometric properties of (S^2, d) are established using the dynamics.

Proposition 4.6. *Let G be a one-ended convergence group on S^2, with a uniform action on its limit set. Let us denote by $\mathcal{P}$ the collection of boundary components of its ordinary set and let us endow Λ_G with a metric from its gauge. The following statements are true.*

(a) *The limit set Λ_G is LLC and doubling.*
(b) *The limit set Λ_G is doubling and porous relatively to $\mathcal{P}$.*

The LLC property is due to Bonk and Kleiner [BK4] and the doubling property to Coornaert [Coo]. This shows in particular that the gauge admits Ahlfors regular metrics assuming the limit set is connected. These properties may be essentially established with the conformal elevator principle [Haï1, Prop. 4.6]:

Proposition 4.7 (Conformal elevator principle). *Let G be a non-elementary hyperbolic group and consider its boundary ∂G endowed with a metric from its gauge. Then there exist definite sizes $r_0 \geq \delta_0 > 0$ and a distortion function η such that, for any $x \in X$, and any $r \in (0, \operatorname{diam} \partial G/2]$, there exists $g \in G$ such that $g(B(x,r)) \supset B(g(x), r_0)$,* $\operatorname{diam} B(g(x), r_0) \geq 2\delta_0$ *and $g|_{B(x,r)}$ is η-quasisymmetric.*

We now check that we may build a sphere quasisymmetrically equivalent to the Riemann sphere, cf. Theorem 2.13.

Proposition 4.8. *Assume (Λ_G, d_Λ) is Ahlfors Q-regular with $Q < 2$. For every component Ω of Ω_G, there exists a metric d_Ω on $\overline{\mathbb{D}}$ in the Ahlfors regular conformal gauge of the closed unit disk such that*

- $(\overline{\mathbb{D}}, d_\Omega)$ *is uniformly* 2*-regular,* $\mathbb{S}^1$ *is uniformly porous and there is a constant $\Delta > 0$ such that* $\operatorname{diam}_{d_\Omega} \overline{\mathbb{D}} \leq \Delta \operatorname{diam}_{d_\Lambda} \partial\Omega$*;*
- *there is a homeomorphism $\varphi_\Omega \colon \overline{\mathbb{D}} \to \overline{\Omega}$ such that*
  (1) *the restriction $\varphi_\Omega \colon (\mathbb{S}^1, d_\Omega) \to (\partial\Omega, d_\Lambda)$ is uniformly bi-Lipschitz;*
  (2) *if Ω and Ω' belong to the same orbit, then there exists $g \in G$ such that $\varphi_{\Omega'} = \varphi_\Omega \circ g$.*

The proof relies on the following Ahlfors-Beurling type theorem (a more precise statement appears in [Haï2]).

Proposition 4.9. *Let (X, d_X) and (Y, d_Y) be connected compact metric spaces. Let us assume that there is an η-quasisymmetric embedding $f \colon Y \to X$ with* $\operatorname{diam}_Y Y = \operatorname{diam}_X f(Y)$*. Then there is a metric $\hat{d}$ on X such that*

(1) $\mathrm{id} \colon (X, d_X) \to (X, \hat{d})$ *is $\widehat{\eta}$-quasisymmetric;*

(2) $f\colon (Y,d_Y)\to (X,\hat d)$ *is bi-Lipschitz onto its image: there exists* $L\ge 1$ *such that, for all* $y_1,y_2\in Y$,
$$\frac{1}{L}d_Y(y_1,y_2)\le \hat d(f(y_1),f(y_2))\le d_Y(y_1,y_2).$$

If we assume that X *and* Y *are* Q_X*- and* Q_Y*-regular respectively with* $Q_Y<Q_X$ *and that* $f(Y)$ *is porous in* X*, then* $\hat d$ *is* Q_X*-regular.*

All the constants involved and $\widehat\eta$ *only depend on* η.

4.3. Characterization of convex-cocompact Kleinian groups. We sketch the proof of Theorem 1.3 in the group case.

If $\Lambda_G=S^2$, then Proposition 4.6 implies with the fact that Λ_G is Ahlfors 2-regular that S^2 is quasisymmetrically equivalent to $\widehat{\mathbb C}$ and that the action is uniformly quasi-Möbius on $\widehat{\mathbb C}$. So the group is Kleinian cocompact, cf. Theorem 2.7.

We now assume that Λ_G is a proper subset of the sphere and we apply the construction made above, thanks to Proposition 4.8. Renormalizing the metrics d_Ω, we may assume that $\varphi_\Omega\colon(\mathbb S^1,d_\Omega)\to(\partial\Omega,d_\Lambda)$ is 1-Lipschitz so we may consider the patchwork metric d obtained from § 2.4.

We check that the assumptions of Theorem 2.13 hold: this follows from Proposition 4.8 together with Proposition 4.6. The patchwork sphere (S^2,d) is quasisymmetric to the Riemann sphere. Let us conjugate the action of G to $\widehat{\mathbb C}$.

From our constructions, we know that the action of G is uniformly quasi-Möbius

- restricted to X and
- restricted to each connected component of $S^2\setminus X$.

Thus, one needs to check that the map is uniformly locally weakly quasisymmetric in S^2 for points on X. To obtain the necessary estimates, we combine the controls obtained in the closures of the components Ω and on X to see that they match to provide uniform global bounds on S^2. We may then conclude the proof by invoking Corollary 2.10. ∎

5. Cxc dynamics. This section is based on [HP1, Haï4]. Let $f\colon S^2\to S^2$ be an orientation-preserving finite branched covering of the sphere of degree $d\ge 2$, and let us assume that $\mathfrak X_1\subset\subset\mathfrak X_0\subset S^2$ are open subsets of the sphere such that $f\colon\mathfrak X_1\to\mathfrak X_0$ is also a finite branched covering of degree d. Let $X=\bigcap f^{-n}(\mathfrak X_0)$ denote the *repellor* of f.

Given a finite cover $\mathcal U$ of X by open connected sets of $\mathfrak X_0$, one defines a sequence of covers $\{\mathcal U_n\}$ by letting $\mathcal U_n$ denote the collection of connected components of $f^{-n}(U)$ for $U\in\mathcal U$.

The map $f\colon\mathfrak X_1\to\mathfrak X_0$ is *topologically coarse expanding conformal (top. cxc)* if there exists a finite covering $\mathcal U$ of X such that

(1) [Irreducibility] For any open set U that intersects X, there is some iterate $n\ge 0$ such that $X\subset f^n(U)$.
(2) [Expansion]
$$\lim_{n\to\infty}\max\{\operatorname{diam}W,\ W\in\mathcal U_n\}=0.$$

(3) [Degree] There is some $d_{max} \in \mathbb{N}$ such that, for all $n \geq 1$ and $W \in \mathcal{U}_n$,

$$\deg(W \xrightarrow{f^n} f^n(W)) \leq d_{max}.$$

Planar finite branched coverings have the following additional property.

Fact 5.1. *Let $f\colon U \to V$ be a finite branched covering where V is a topological disk and such that V has at most one singular value. Then each connected component W of U is simply connected and there is a point $w \in W$ such that $d_f(w) = \deg(W \xrightarrow{f} V)$.*

It follows from the above fact that we may choose a finite cover $\mathcal{U}$ defining the cxc property that also satisfies the following condition:

[Planar] For all $n \geq 1$ and all $W \in \mathcal{U}_n$, W is simply connected and $\overline{W}$ contains at most one critical value, and, in this case, it belongs to $W \cap X$.

Theorem 3.5 proves the following:

Theorem 5.2 (Canonical Gauge). *Given a topologically cxc map as above, we may endow X with a distance d_v with the following properties.*

(1) *There exist constants $\theta \in (0,1)$ and $r_0 > 0$ such that, for any $x \in X$ and any iterate $k \geq 1$, $f^k(B_v(x, r\theta^k)) = B_v(f^k(x), r)$ for any $r < r_0$.*
(2) *For any $n \geq 1$ and any $W \in \mathcal{U}_n$, there is a point $x \in W$ such that $(W \cap X) \asymp B_v(x, \theta^n)$.*
(3) *The metric is Ahlfors regular of dimension $\log d / \log(1/\theta)$ and the map is absolutely continuous with respect to the corresponding Hausdorff measure.*

Furthermore, if d is another metric sharing these properties, then there is some positive number $\alpha > 0$ such that $d \asymp d_v^\alpha$. In particular, the identity map between (X, d_v) and (X, d) is quasisymmetric.

Metrics that satisfy the conclusions of Theorem 5.2 are called *visual distances.*

We recall [HP1, Prop. 3.3.2 and 3.3.3] which provides us with some control on the distortion of the dynamics.

Proposition 5.3. *Fix $(f, \mathfrak{X}_1, \mathfrak{X}_0, X)$ and we assume that X is endowed with a visual metric.*

(1) *There is some constant $C > 1$ such that, for any level $n \geq 0$ and any $W \in \mathcal{U}_n$, there is a point $\xi \in X$ so that*

$$B_v(\xi, (1/C)\theta^n) \subset W \cap X \subset B_v(\xi, C\theta^n).$$

(2) *A maximal radius $r_0 > 0$ exists such that, for any $r \in (0, r_0)$ and any $\xi \in X$, there exist levels $n, m \geq 0$, $W \in \mathcal{U}_n$ and $W' \in \mathcal{U}_m$ such that $|n - m| = O(1)$,*

$$W' \cap X \subset B_v(\xi, r) \subset W,$$

and

$$\max\{\mathrm{Round}(W \cap X, \xi), \mathrm{Round}(W' \cap X, \xi)\} = O(1).$$

5.1. The topology of X. We start by studying the global structure of the repellor which is built in the definition. Let $\mathcal{C}(X)$ denote the set of connected components of $\widehat{\mathbb{C}}\setminus X$.

PROPOSITION 5.4. *Either X has no interior, or $X = S^2$. In the former case, the elements of $\mathcal{C}(X)$ are all preperiodic and form a finite set of orbits.*

We now give the main dynamical properties of such maps.

THEOREM 5.5. *Let $f\colon \mathfrak{X}_1 \to \mathfrak{X}_0$ be a planar top. cxc map with connected repellor $X \neq S^2$. Then X is locally connected, and, for any connected component Ω of $\widehat{\mathbb{C}} \setminus X$, there exists a continuous map $p_\Omega\colon \mathbb{S}^1 \to \partial\Omega$, such that, for any $x \in \mathbb{S}^1$, $f \circ p_\Omega(x) = p_\Omega(x^{d_\Omega})$ where $d_\Omega = \deg(\Omega \xrightarrow{f} f(\Omega))$.*

The local connectedness is established from Whyburn's criterion [Why, Thm. VI.4.4]: one shows that (a) the components $\mathcal{C}(X)$ form a *null sequence*, i.e., given any metric compatible with the topology of the sphere, only finitely many components have diameter at least δ for any $\delta > 0$, and (b) each component has locally connected boundary. The former follows from the [Expansion] assumption and the latter from the existence of the maps p_Ω. The existence of such maps is first established for the periodic components, and then spread out by the dynamics. Given a periodic component Ω of period k, we combine the [Expansion] axiom with a pull-back argument a la Douady-Hubbard to show the uniform convergence of maps $p_n\colon \mathbb{S}^1 \to \Omega$ that satisfy $f^k \circ p_{n+1}(z) = p_n(z^{d_\Omega})$ where $d_\Omega = \deg(\Omega \xrightarrow{f^k} \Omega)$ and $z \in \mathbb{S}^1$ (cf. [DH, Prop. III.3 and III.4]).

5.2. The geometry of X. We now state the geometric properties of the repellor that will enable us to construct a metric sphere. The LLC property will follow once we construct the sphere and extend the dynamics.

THEOREM 5.6. *Let $f\colon \mathfrak{X}_1 \to \mathfrak{X}_0$ be a planar top. cxc map with connected repellor X. Let us endow X with a visual distance d_v The following properties hold.*

(1) *The space X is (BT), doubling and porous relative to $\mathcal{C}(X)$.*
(2) *There exist a power $\alpha \in (0,1)$ and constants σ_Ω for each $\Omega \in \mathcal{C}(X)$ so that $p_\Omega\colon (\mathbb{S}^1, \sigma_\Omega d_e^\alpha) \to (\partial\Omega, d_v)$ are surjective 1-Lipschitz uniformly regular maps and the boundaries $\partial\Omega$ are uniformly (BT), where d_e stands for the Euclidean metric.*

Since the repellor is known to be locally connected, one obtains the bounded turning property by picking suitably the initial cover. The relative properties follow from a combination of the [Expansion] and the [Degree] axioms. The proofs of the uniform regularity and of the uniform (BT) conditions are more delicate to establish. One difficulty comes from the fact that the restriction of the dynamics to the boundary of a component need not be a finite branched covering so the effect of pulling back components is not that clear. We first deal with periodic components, then with preperiodic components that contain critical orbits (there are only finitely many of them). We obtain uniform bounds for all the rest thanks to Lemma 2.16.

5.3. Construction of a sphere and extension of the dynamics. Let $f\colon \mathfrak{X}_1 \to \mathfrak{X}_0$ be a planar topological cxc map with connected repellor X and let us assume that $\mathcal{U}$ is a finite cover by disks that satisfies the axioms [Irred], [Exp], [Deg] and [Planar]. We consider a visual metric d_v on X given by Theorem 5.2. According to Theorem 5.6, X has the bounded turning property, and, for any connected component Ω of $\widehat{\mathbb{C}} \setminus X$,

(1) the boundary $\partial\Omega$ is uniformly (BT) and
(2) there exists a 1-Lipschitz regular map $p_\Omega \colon \mathbb{S}^1 \to \partial\Omega$ when $\mathbb{S}^1$ is equipped with an appropriate metric d_Ω of its gauge, such that, for any $x \in \mathbb{S}^1$, $f \circ p_\Omega(x) = p_\Omega(x^{d_\Omega})$ where $d_\Omega = \deg(\Omega \xrightarrow{f} f(\Omega))$.

Let us extend d_Ω using Proposition 4.9 so that $(\overline{\mathbb{D}}, d_\Omega)$ is uniformly locally Ahlfors 2-regular. We may then construct the patchwork metric, see § 2.4. We conclude from above that X is porous in S^2 and S^2 is a doubling and (BT) metric space. We write (Σ, d_Σ) for this new metric sphere.

We extend $f \colon X \to X$ to $F \colon \Sigma \to \Sigma$ as follows. Let Ω be a component of $S^2 \setminus X$; since for $x \in \mathbb{S}^1$, $f \circ p_\Omega(x) = p_\Omega(x^{d_\Omega})$ holds, where $d_\Omega = \deg(\Omega \xrightarrow{f} f(\Omega))$, we set $F(z) = z^{d_\Omega}$ for $z \in \Omega$ (identified to $\mathbb{D}$ via the homeomorphism between $\overline{\Omega}$ and $\overline{\mathbb{D}}/p_\Omega$).

We note that F is a well-defined finite branched covering of the sphere since X is an E-set i.e, for any $\delta > 0$, $\mathcal{C}(X)$ has only finitely many component of diameter at least δ.

We obtain the following properties.

Proposition 5.7. *There exists a cover $\mathcal{V}$ of X in Σ so that the map F is top. cxc (including the [Planar] axiom) and satisfies the following properties.*

- [Diam] *For any $V \in \mathcal{V}_n$,* $\operatorname{diam} V_n \asymp \theta^n$.
- [Round] *For any K, there exists K' such that, for $n, k \geq 0$ $V \in \mathcal{V}_{n+k}$ and $x \in V$, if* $\operatorname{Round}(V, x) \leq K$ *then* $\operatorname{Round}(f^k(V), f^k(x)) \leq K'$ *and if* $\operatorname{Round}(f^k(V), f^k(x)) \leq K$ *then* $\operatorname{Round}(V, x) \leq K'$.

This implies the LLC property since elements of $\mathcal{V}_n$ are Jordan domains that behave like balls of radius θ^n in the patchwork metric. Moreover, we may also deduce that F is uniformly quasiregular in a metric sense.

Corollary 5.8. *We have the following properties.*

(1) *The sphere (Σ, d_Σ) is LLC, doubling, $X \hookrightarrow (S^2, d)$ is porous, $d|_X = d_v$ and there is some constant $c > 0$ such that, for any $x \notin X$ and $r < cd(x, X)$, $\mathcal{H}_2(B(x, r)) \asymp r^2$.*
(2) *The map F is uniformly quasiregular and $F|_X = f|_X$.*

5.4. Characterization of semi-hyperbolic rational maps. We now want to conclude the proof of Theorem 1.3 in the cxc setting. First, if $X = S^2$, then covering as above S^2 with Jordan domains, we obtain the LLC property. Thus, we may conclude with Corollary 2.10 by picking an Ahlfors 2-regular metric in its gauge.

If X is a proper subset of S^2, then we have shown above how to construct a metric sphere (Σ, d_Σ) that is LLC together with some uniformly quasiregular dynamics F extending f, cf. Corollary 5.8. We further know that the sphere is doubling and that X is a porous subset. But, since we started with a visual distance, there is no reason why the dimension of X should be less than two.

Thus, let us consider an Ahlfors Q-regular metric d_X in X in the gauge of f with $Q < 2$. We may apply Proposition 4.9 to $(X, d_X) \hookrightarrow \Sigma$. We note that since d_Σ was locally

Ahlfors 2-regular in $\Sigma \setminus X$ and X is Q-regular and porous, we may conclude that this new sphere is LLC and 2-regular, hence quasisymmetric to $\mathbb{S}^2$ according to Theorem 2.9. Thus, F is conjugate to a rational map since it is uniformly quasiregular, cf. Theorem 2.7.

Acknowledgments. I am grateful to Feliks Przytycki for his invitation to give this series of lectures in the context of the Simons Semester in Banach Center "Dynamical systems". I would like to thank the referee for his comments which helped improve the text. This work was partially supported by the grant 346300 for IMPAN from the Simons Foundation and the matching 2015-2019 Polish MNiSW fund and by the ANR projects "GDSous/GSG" no. 12-BS01-0003-01 and "Lambda" no. 13-BS01-0002.

References

[Ahl] L. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand, 1966.

[AIM] K. Astala, T. Iwaniec, and G. Martin, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, Princeton Mathematical Series 48, Princeton Univ. Press, 2009.

[BA] A. Beurling and L. Ahlfors, *The boundary correspondence under quasiconformal mappings*, Acta Math. 96 (1956), 125–142.

[BK1] M. Bonk and B. Kleiner, *Quasisymmetric parametrizations of two-dimensional metric spheres*, Invent. Math. 150 (2002), 127–183.

[BK2] M. Bonk and B. Kleiner, *Rigidity for quasi-Möbius group actions*, J. Differential Geom. 61 (2002), 81–106.

[BK3] M. Bonk and B. Kleiner, *Conformal dimension and Gromov hyperbolic groups with 2-sphere boundary*, Geom. Topol. 9 (2005), 219–246.

[BK4] M. Bonk and B. Kleiner, *Quasi-hyperbolic planes in hyperbolic groups*, Proc. Amer. Math. Soc. 133 (2005), 2491–2494.

[BS] M. Bonk and O. Schramm, *Embeddings of Gromov hyperbolic spaces*, Geom. Funct. Anal. 10 (2000), 266–306.

[BP] M. Bourdon and H. Pajot, *Cohomologie ℓ_p et espaces de Besov*, J. Reine Angew. Math. 558 (2003), 85–108.

[Bow1] B. H. Bowditch, *A topological characterisation of hyperbolic groups*, J. Amer. Math. Soc. 11 (1998), 643–667.

[Bow2] B. H. Bowditch, *Convergence groups and configuration spaces*, in: Geometric Group Theory down under (Canberra, 1996), de Gruyter, 1999, 23–54.

[Bow3] B. H. Bowditch, *Uniform hyperbolicity of the curve graphs*, Pacific J. Math. 269 (2014), 269–280.

[BH] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-positive Curvature*, Grundlehren Math. Wiss. 319, Springer, 1999.

[BBI] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, Grad. Studies in Math. 37, Amer. Math. Soc., 2001.

[CG] L. Carleson and T. W. Gamelin, *Complex Dynamics*, Universitext, Springer, 1993.

[CJY] L. Carleson, P. W. Jones, and J.-C. Yoccoz, *Julia and John*, Bol. Soc. Brasil. Mat. (N.S.) 25 (1994), 1–30.

[CP] M. Carrasco Piaggio, *On the conformal gauge of a compact metric space*, Ann. Sci. École Norm. Sup. (4) 46 (2013), 495–548.

[Coo] M. Coornaert, *Mesures de Patterson–Sullivan sur le bord d'un espace hyperbolique au sens de Gromov*, Pacific J. Math. 159 (1993), 241–270.

[DS] G. David and S. Semmes, *Fractured Fractals and Broken Dreams*, Oxford Lecture Ser. Math. Appl. 7, Oxford Univ. Press, 1997.

[DH] A. Douady and J. Hubbard, *Étude dynamique des polynômes complexes, I, II*, Publications Mathématiques d'Orsay 1984/5.

[Geh] F. W. Gehring, *The definitions and exceptional sets for quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I No. 281 (1960), 28.

[GM] F. W. Gehring and G. J. Martin, *Discrete quasiconformal groups. I*, Proc. London Math. Soc. (3) 55 (1987), 331–358.

[Ger1] V. Gerasimov, *Expansive convergence groups are relatively hyperbolic*, Geom. Funct. Anal. 19 (2009), 137–169.

[Ger2] V. Gerasimov, *Floyd maps for relatively hyperbolic groups*, Geom. Funct. Anal. 22 (2012), 1361–1399.

[GP] V. Gerasimov and L. Potyagailo, *Quasi-isometric maps and Floyd boundaries of relatively hyperbolic groups*, J. Eur. Math. Soc. 15 (2013), 2115–2137.

[GdlH] É. Ghys and P. de la Harpe (eds.), *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progr. Math. 83, Birkhäuser Boston Inc., 1990.

[Gro] M. Gromov, *Hyperbolic groups*, in: Essays in Group Theory, Math. Sci. Res. Inst. Publ. 8, Springer, 1987, 75–263.

[Haï1] P. Haïssinsky, *Géométrie quasiconforme, analyse au bord des espaces métriques hyperboliques et rigidités [d'après Mostow, Pansu, Bourdon, Pajot, Bonk, Kleiner...]*, in: Séminaire Bourbaki, Vol. 2007/2008, Astérisque 326 (2009), exp. no. 993, ix, 321–362.

[Haï2] P. Haïssinsky, *A sewing problem in metric spaces*, Ann. Acad. Sci. Fenn. Math. 34 (2009), 319–345.

[Haï3] P. Haïssinsky, *Hyperbolic groups with planar boundaries*, Invent. Math. 201 (2015), 239–307.

[Haï4] P. Haïssinsky, *Cxc maps with planar repellors of low conformal dimension*, in preparation.

[HP1] P. Haïssinsky and K. M. Pilgrim, *Coarse expanding conformal dynamics*, Astérisque 325 (2009), viii+139 pp.

[HP2] P. Haïssinsky and K. M. Pilgrim, *Minimal Ahlfors regular conformal dimension of coarse expanding conformal dynamics on the sphere*, Duke Math. J. 163 (2014), 2517–2559.

[Hei] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Universitext, Springer, 2001.

[Hin] A. Hinkkanen, *Uniformly quasiregular semigroups in two dimensions*, Ann. Acad. Sci. Fenn. Math. 21 (1996), 205–222.

[KK] S. Kallunki and P. Koskela, *Metric definition of μ-homeomorphisms*, Michigan Math. J. 51 (2003), 141–151.

[KB] I. Kapovich and N. Benakli, *Boundaries of hyperbolic groups*, in: Combinatorial and Geometric Group Theory (New York, 2000/Hoboken, NJ, 2001), Contemp. Math. 296, Amer. Math. Soc., 2002, 39–93.

[KR] P. Koskela and S. Rogovin, *Linear dilatation and absolute continuity*, Ann. Acad. Sci. Fenn. Math. 30 (2005), 385–392.

[LV] O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, Grundlehren Math. Wiss. 126, Springer, 1973.

[MT] J. M. Mackay and J. T. Tyson, *Conformal Dimension*, Univ. Lecture Ser. 54, Amer. Math. Soc., 2010.

[Mar] A. Marden, *Outer Circles*, Cambridge Univ. Press, 2007.

[Mat] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge Stud. Adv. Math. 44, Cambridge Univ. Press, 1995.

[Mil] J. Milnor, *Dynamics in One Complex Variable*, Ann. of Math. Stud. 160, Princeton Univ. Press, 2006.

[Pan] P. Pansu, *Dimension conforme et sphère à l'infini des variétés à courbure négative*, Ann. Acad. Sci. Fenn. Ser. A I Math. 14 (1989), 177–212.

[Pau] F. Paulin, *Un groupe hyperbolique est déterminé par son bord*, J. London Math. Soc. (2) 54 (1996), 50–74.

[Ser] C. Series, *A crash course on Kleinian groups*, Rend. Istit. Mat. Univ. Trieste 37 (2005), 1–38 (2006).

[Sul1] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Inst. Hautes Études Sci. Publ. Math. 50 (1979), 171–202.

[Sul2] D. Sullivan, *On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions*, in: Riemann Surfaces and Related Topics (Stony Brook, NY, 1978), Ann. of Math. Stud. 97, Princeton Univ. Press, 1981, 465–496.

[Sul3] D. Sullivan, *Seminar on hyperbolic geometry and conformal dynamical systems*, preprint IHES, 1982.

[Sul4] D. Sullivan, *Conformal dynamical systems*, in: Geometric Dynamics, Lecture Notes in Math. 1007, Springer, 1983.

[Swa] G. A. Swarup, *On the cut point conjecture*, Electron. Res. Announc. Amer. Math. Soc. 2 (1996), 98–100.

[Tuk1] P. Tukia, *On quasiconformal groups*, J. Analyse Math. 46 (1986), 318–346.

[Tuk2] P. Tukia, *Convergence groups and Gromov's metric hyperbolic spaces*, New Zealand J. Math. 23 (1994), 157–187.

[TV] P. Tukia and J. Väisälä, *Quasisymmetric embeddings of metric spaces*, Ann. Acad. Sci. Fenn. Ser. A I Math. 5 (1980), 97–114.

[Väi1] J. Väisälä, *Lectures on n-dimensional Quasiconformal Mappings*, Lecture Notes in Math. 229, Springer, 1971.

[Väi2] J. Väisälä, *Quasi-Möbius maps*, J. Analyse Math. 44 (1984/85), 218–234.

[Why] G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Colloq. Publ. 28, Amer. Math. Soc., 1963.